ON A PROPERTY OF GAMMA DISTRIBUTIONS

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Abstract

This paper concerns the stochastic equation $X \stackrel{\mathcal{L}}{=} B(X + C)$, where B, X and C are independent. This equation has appeared in a number of contexts, notably in actuarial science. An apparently new property of gamma variables (Theorem 1) leads to the derivation of a new explicit example of solution of the stochastic equation (Theorem 3), where B is the product of two independent beta variables, C is gamma and X is the product of independent beta and gamma variables. It is also shown that a number of previously known explicit examples are direct algebraic consequences of a well-known property of gamma variables (Corollary 2).

DISCOUNTED SUMS; HYPERGEOMETRIC FUNCTIONS; GAMMA VARIABLES

1. Introduction

Suppose $\{B_n, n \geq 1\}$ and $\{C_n, n \geq 0\}$ are two independent i.i.d. sequences, and consider the stochastic difference equation

$$X_{n+1} = B_{n+1}(X_n + C_n) \tag{1}$$

where $X_0 = x_0$ is a constant. Iterating (1) we get

$$X_n = x_0 B_1 \cdots B_n + \sum_{k=0}^{n-1} C_k B_{k+1} \cdots B_n. \tag{2}$$

$\{X_n\}$ is a homogeneous Markov chain. A related process is

$$Y_n = \sum_{k=1}^{n} C_k B_1 \cdots B_k. \tag{3}$$

$\{Y_n\}$ is not a Markov chain, but it can be seen that, given $x_0 = 0$, X_n and Y_n have the same distribution for each fixed $n \geq 1$ (simply reverse the order of the indices of the B's and C's, and use the independence assumption).

Equations such as (1), (2) or (3) arise in a number of contexts (see Vervaat, 1979, for examples). In actuarial science, X_n might represent the accumulated value of amounts $\{C_0, C_1, \ldots, C_{n-1}\}$, when the accumulating factors (i.e. one plus the rate of return) are $\{B_1, B_2, \ldots, B_n\}$. Dufresne (1991) describes the actuarial applications and also gives formulas for the moments X_n and Y_n.

Vervaat (1979) states the following sufficient conditions for the existence and uniqueness of the limit distribution of X_n as $n \to \infty$:

$$\mathsf{E}(\log B_1) \;<\; 0, \quad \mathsf{E}(\log |C_1|)_+ \;<\; \infty. \tag{4}$$

The same conditions ensure the almost sure convergence of Y_n. When X_n converges in law the limit X must satisfy

$$X \stackrel{\mathcal{L}}{=} B(X + C), \qquad B,\, X \text{ and } C \text{ independent.} \tag{5}$$

A number of explicit examples of solutions of (5) have been found, see Vervaat (1979) and Chamayou and Letac (1991). Embrechts and Goldie (1994) provide further results on the convergence of X_n and Y_n.

This paper has two goals. First, to derive a new explicit solution of (5), based on a certain property of gamma variables (Theorem 1). The law of X turns out to be the product of independent beta and gamma distributions. The moment generating function of this type of distribution is given in Section 2. The new example is derived in Section 4. The second goal (Section 5) is to indicate how several of the known explicit examples of (5) may be given simple proofs based on a very well known consequence of Theorem 1 (Corollary 2). These new proofs are "algebraic", in the sense that they entirely rely on transformations of the identity contained in Corollary 2. This is in contrast with earlier proofs, which used *ad hoc* differential equations or Mellin transforms arguments. The resulting theory is more unified and may lead to further explicit examples.

2. The product of a beta and a gamma distribution

Notation. The variable G_a has a $\Gamma(a, 1)$ distribution. Primes (and numerals if necessary) will be used to indicate that two or more gamma variables are independent. If $V_i \sim \mathcal{L}_i$, $i = 1, 2$, are independent then the distribution of their product

$U = V_1 V_2$ will be denoted $\mathcal{L}_1 \odot \mathcal{L}_2$. The distribution of their sum will be denoted $\mathcal{L}_1 * \mathcal{L}_2$.

B has a beta distribution of the first kind with parameters a and b $(a, b > 0)$ if its density is

$$f_B(x) \;=\; \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, x^{a-1}(1-x)^{b-1}\mathbf{1}_{(0,1)}(x).$$

This will be denoted $B \sim \beta^{(1)}_{a,b}$. In this paper all gamma distributions have 1 as scale parameter: given $c > 0$, we write $G \sim \Gamma(c,1)$ if G has density

$$f_G(x) \;=\; \Gamma(c)^{-1} x^{c-1} e^{-x}\mathbf{1}_{(0,\infty)}(x).$$

Suppose $X \sim \beta^{(1)}_{a,b} \odot \Gamma(c,1)$. Then

$$\begin{aligned} \mathsf{E}e^{tX} &= \mathsf{E}(1-tB)^{-c}, \qquad B \sim \beta^{(1)}_{a,b} \\ &= F(a, c; a+b; t), \qquad t < 1, \end{aligned}$$

where ($z \in \mathbf{C}$, $\mathrm{Re}\,\zeta > \mathrm{Re}\,\gamma > 0$)

$$\begin{aligned} F(\alpha,\gamma;\zeta;z) &= \int_0^1 \frac{\Gamma(\zeta)}{\Gamma(\gamma)\Gamma(\zeta-\gamma)}\, t^{\gamma-1}(1-t)^{\zeta-\gamma-1}(1-tz)^{-\alpha}dt, \qquad |\arg(1-z)| < \pi \\ &= \sum_{k=0}^{\infty} \frac{(\alpha)_k(\gamma)_k}{(\zeta)_k}\,\frac{z^k}{k!}, \qquad |z| < 1 \\ &\quad (\alpha)_0 = 1, \quad (\alpha)_k = (\alpha+k-1)(\alpha)_{k-1}, \quad k > 0. \end{aligned}$$

$F(\alpha,\gamma;\zeta;z)$ is known as the hypergeometric function (see Chapter 9 of Lebedev, 1972). Observe that $F(\alpha,\gamma;\zeta;z) = F(\gamma,\alpha;\zeta;z)$, which implies

$$\beta^{(1)}_{a,b} \odot \Gamma(c,1) \;=\; \beta^{(1)}_{c,a+b-c} \odot \Gamma(a,1), \qquad c < a+b.$$

Another way of stating this is: for any $a, b, c > 0$

$$\frac{G_a}{G_a + G'_b + G''_c}\cdot G'''_b \;\stackrel{\mathcal{L}}{=}\; \frac{G_b}{G'_a + G_b + G''_c}\cdot G'''_a. \tag{6}$$

3. Two properties of beta and gamma distributions

Theorem 1. *For any $a, b, c > 0$*

$$\frac{G_a}{G_a + G'_{b+c}} \cdot G''_b + G'''_c \overset{\mathcal{L}}{=} \frac{G_{b+c}}{G'_a + G_{b+c}} \cdot G''_{a+c}.$$

PROOF. From Section 2 the m.g.f. of the variable on the right is $F(b+c, a+c; a+b+c; t)$, $t < 1$. Using the identity

$$F(\alpha,\gamma;\zeta;z) \;=\; (1-z)^{\zeta-\alpha-\gamma}F(\zeta-\alpha,\zeta-\gamma;\zeta;z), \qquad |\arg(1-z)| < \pi$$

(Lebedev, 1972, p. 248) we get

$$F(b+c, a+c; a+b+c; t) \;=\; (1-t)^{-c}F(a, b; a+b+c; t), \qquad t < 1. \qquad \square$$

Letting $b \to 0$ in Theorem 1 we obtain:

Corollary 2. *For any $a, b > 0$*

$$\frac{G_a}{G_a + G'_b} \cdot (G''_a + G'''_b) \overset{\mathcal{L}}{=} G_a. \tag{7}$$

This result is also the consequence of the (familiar) independence of $Y_1 = G_a/(G_a + G_b)$ and $Y_2 = G_a + G_b$, see for example Hogg and Craig (1978, pp. 138-139, and Exercise 4.43, p. 153) or Hoel *et al* (1971, Exercise 41, p.172). The distribution of Y_1Y_2 is then clearly the same as the distribution of either side of the Eq. (7). The independence of Y_1 and Y_2 characterizes the gamma distribution, see Lukacs (1955).

4. A new explicit example

Theorem 3. *Suppose $B \sim \beta^{(1)}_{a,c} \odot \beta^{(1)}_{b,c}$ and $C \sim \Gamma(c, 1)$. Then Eq. (5) has unique solution*

$$X \sim \beta^{(1)}_{a,b+c} \odot \Gamma(b, 1).$$

Proof. Conditions (4) are obviously satisfied. Theorem 1 says that

$$X + C \overset{\mathcal{L}}{=} \frac{G_b + G'_c}{G''_a + G_b + G'_c} \cdot G'''_{a+c}. \tag{8}$$

Then

$$B(X + C) \overset{\mathcal{L}}{=} \frac{G^{(4)}_a}{G^{(4)}_a + G^{(5)}_c} \cdot \frac{G^{(6)}_b}{G^{(6)}_b + G^{(7)}_c} \cdot \frac{G_b + G'_c}{G''_a + G_b + G'_c} \cdot G'''_{a+c}.$$

There are four factors in the expression on the right. By Corollary 2 the first and fourth factors may be replaced by $G^{(8)}_a$. As to the second and third factors define $f_1(x, y) = x/(x + y)$, $f_2(x, y) = x + y$, $U = (G_b, G'_c)$, $U' = (G^{(6)}_b, G^{(7)}_c)$, and $g(f_1, f_2, v) = f_1 f_2/(v + f_2)$. The variables $\{f_1(U), f_2(U), G''_a\}$ are independent and so

$$g(f_1(U'), f_2(U), G''_a) \overset{\mathcal{L}}{=} g(f_1(U), f_2(U), G''_a) = \frac{G_b}{G''_a + G_b + G'_c}. \tag{9}$$

This, together with (6), implies

$$B(X + C) \overset{\mathcal{L}}{=} \frac{G''_a}{G''_a + G_b + G'_c} \cdot G'''_b \overset{\mathcal{L}}{=} X. \qquad \square$$

Remarks 1. Given Eq. (8) the proof of Theorem 3 may also be completed using the Mellin transform $X \mapsto \mathsf{E}X^t$. The above proof shows that the underlying "algebraic structure" (given in Theorem 1) is nearly sufficient to obtain Theorem 3; the only other fact needed is identity (6).

2. As pointed out in the proof the law of X may also be expressed as $\beta^{(1)}_{b,a+c} \odot \Gamma(a, 1)$. The Mellin transform of $A \sim \beta^{(1)}_{a,b}$ being

$$\mathsf{E}A^t = \frac{\Gamma(a + b)}{\Gamma(a + b + t)} \frac{\Gamma(a + t)}{\Gamma(a)},$$

it can be seen that the law of B is also $\beta^{(1)}_{a,b+c-a} \odot \beta^{(1)}_{b,a+c-b}$. $\square$

Corollary 4. *Suppose $B \sim \beta^{(1)}_{a,2c}$ and $C \sim \Gamma(c, 1)$. Then Eq. (5) has unique solution*

$$X \sim \beta^{(1)}_{a+c,a+c} \odot \Gamma(a, 1) = \beta^{(1)}_{a,a+2c} \odot \Gamma(a + c, 1).$$

PROOF. Let $b = a'$ and $a = a' + c$ in Theorem 2, then proceed as in (9) to verify that

$$B \sim \beta^{(1)}_{a+c,c} \odot \beta^{(1)}_{a,c} \;=\; \beta^{(1)}_{a,2c}. \qquad \square$$

5. A unified treatment of some known examples

This section shows that three known examples of explicit solutions of (5) are in fact direct algebraic consequences of Corollary 2. These examples had previously been given proofs based on differential equations, Mellin transforms or Laplace transforms.

Corollary 5. *Suppose* $B \sim \beta^{(1)}_{a,b}$, $C \sim \Gamma(b,1)$. *Then Eq. (5) has unique solution* $X \sim \Gamma(a,1)$.

PROOF. Immediate from Corollary 2. $\square$

This example was obtained by Letac (1986). The particular case $b = 1$ had previously been derived by various authors (see Dufresne, 1991, and Vervaat, 1979, for references). Observe that this example may also be obtained by letting $b \to \infty$ in Theorem 3.

The second and third examples are due to Chamayou and Letac (1991). X has a beta distribution of the second kind with parameters a and b ($a, b > 0$), denoted $X \sim \beta^{(2)}_{a,b}$, if it has density

$$f_X(x) \;=\; \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, x^{a-1}(1+x)^{-a-b}\mathbf{1}_{(0,\infty)}(x).$$

It can be verified that $G_a/G'_b \sim \beta^{(2)}_{a,b}$ (this family thus includes all F distributions).

Corollary 6. *Suppose* $B \sim \beta^{(2)}_{a,a+b}$, $C \equiv 1$. *Then Eq. (5) has unique solution* $X \sim \beta^{(2)}_{a,b}$.

Proof. Reverse the roles of a and b in Corollary 2, take reciprocals and then multiply by $G_a^{(4)}$ to get

$$\begin{aligned} \frac{1}{G_b} &\overset{\mathcal{L}}{=} \left(1+\frac{G_a}{G'_b}\right)\frac{1}{G''_a+G'''_b} \\ \frac{G_a^{(4)}}{G_b} &\overset{\mathcal{L}}{=} \left(1+\frac{G_a}{G'_b}\right)\frac{G_a^{(4)}}{G''_a+G'''_b} \end{aligned}$$

or $\beta^{(2)}_{a,b} = (\delta_1 * \beta^{(2)}_{a,b}) \odot \beta^{(2)}_{a,a+b}$, where δ_y is the Dirac measure concentrated on the point y. □

Corollary 7. *Suppose* $-B \sim \beta^{(1)}_{a,b}$, $C \equiv -1$. *Then Eq. (5) has unique solution* $X \sim \beta^{(1)}_{a,a+b}$.

Proof. From Corollary 2

$$\begin{aligned} X \overset{\mathcal{L}}{=} \frac{G_a}{G_a+G'_b+G''_a} &= \frac{G_a}{G_a+G'_b}\cdot\frac{G_a+G'_b}{G_a+G'_b+G''_a} \\ &= \frac{G_a}{G_a+G'_b}\cdot\frac{1}{1+\frac{G''_a}{G_a+G'_b}} \\ &\overset{\mathcal{L}}{=} \frac{G'''_a}{G'''_a+G_b^{(4)}}\cdot\frac{G_a+G'_b}{G_a+G'_b+G''_a} \\ &= -\frac{G'''_a}{G'''_a+G_b^{(4)}}\cdot\left(\frac{G''_a}{G_a+G'_b+G''_a}-1\right). \end{aligned}$$

□

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