A GENERAL CLASS OF RISK MODELS

Abstract

We consider the actuarial risk model when the waiting times or claims have a Laplace transform which is a rational function. This generalizes the classical model, where the waiting times are exponential, and gives more flexibility in the modelling of a risk business. Ruin is seen as a random walk crossing a barrier; the summands of the random walk are expressed as the difference of the waiting time and the claim. The class $\mathcal{R}^f$ of distributions which have finite rational Laplace transforms includes the so-called phase-type distributions. For waiting times in $\mathcal{R}^f$, the Laplace transform of the ruin probability is obtained explicitly; if the claims are in $\mathcal{R}^f$, then the probability of ruin is a combination of exponentials times polynomials, which can be found in closed-form.

Risk theory; Renewal process; Random walk

1. Introduction

There is an abundant literature on the classical insurance no-interest risk model, which assumes that the waiting times between claims are exponential (arrival times thus forming a Poisson process). Fundamental to this model is the integral equation

$$\psi'(u) \;=\; \frac{\lambda}{c}\left\{\psi(u) - \int_0^u \psi(u-x)dF_X(x) - [1 - F_X(u)]\right\},$$

where $\psi(u)$ is the probability of ruin, $1/\lambda$ is the mean waiting times, c is the rate at which premiums are received, and F_X is the distribution function of the claims. The Laplace transform of the probability of ruin is easily obtained from this equation. Several numerical techniques are available for the numerical inversion of this transform; however, up to now its explicit inversion has only been done in special cases, *e.g.* if the claims distribution is a combination of exponentials (Gerber, 1979, p.118). Sparre Andersen (1957) suggested a model where the i.i.d. waiting times have an arbitrary distribution, and derived integral equation (2.1) below. Since then, the theories of random walks and of queues (see for instance Spitzer (1964)) have provided a more general framework, which has led to explicit results in cases where waiting times or claims have distributions related to the Erlang (for instance, see Borovkov (1976)). Dickson (1998) and Dickson & Hipp (1998, 2000) look at the cases where the waiting times have a **Gamma**$(2, \lambda)$ distribution or a combination of two exponentials. They obtain an explicit expression for the Laplace transform of the probability of ruin by solving a second-order differential equation, itself a consequence of the Sparre Andersen integral equation for this specific case. This approach is rather cumbersome, and appears very difficult as soon as the waiting times are a combination of more than two exponentials, or if they are **Gamma**(n, λ) for an integer n larger than two. By contrast, in this paper we proceed directly with the Laplace transform of integral equation (2.1); this permits the derivation of the Laplace transform of the probability of non-ruin for a wider class of waiting times distributions. This class includes finite combinations of Erlang distributions and other distributions which have a rational Laplace transform. The simplifications brought about by the use of transforms and complex variables in Risk Theory are similar to those the same tools bring to ordinary differential equations with constant coefficients.

The approach we take is to express ruin as a random walk crossing a barrier. This question has been studied by many authors, in Probability as well as in Queueing Theory. Three decades ago, Feller (1971, p.389) already remarked "The literature is vast and bewildering", in connexion with these problems. This paper may not give new answers, but it expresses known ones in a different guise, precisely aimed at calculating the probability of ruin. The results given in Feller (1971, Chapter 12), Borovkov (1976), Asmussen (1992) and elsewhere on random walks can be applied to Risk Theory. However, this paper takes up the problem from scratch, applying elementary results from the theory of functions of a complex variable, and proceeds directly to the ruin problem. The distributions considered include the phase-type distributions, but differential equations are unnecessary in our approach. Ladder variables (Feller, 1971; Bowers *et al.*, 1997) do not appear explicitly, but factorization identities play an essential role, as they do in the so-called "Wiener-Hopf" treatment of random walks (regarding the latter, see the comments in Feller (1971)). Contrary to common usage, we use Laplace transforms, rather than Fourier transforms.

Section 2 states the assumptions underlying the model. Section 3 gives some general properties of the Laplace transform of the survival (or non-ruin) probability, including a lower bound for the probability of ruin, and the removal of zero claims or zero waiting times. Section 4 defines the class of finite rational distributions $\mathcal{R}^f$, and gives examples. Section 5 derives an explicit expression for the Laplace transform of the probability of non-ruin (or "survival probability"), when waiting times are in $\mathcal{R}^f$, the claims distribution being arbitrary. Section 6 derives an expression for the ruin probability when claims are in $\mathcal{R}^f$. Finally, Section 7 concludes the paper and comments on various extensions of the results.

2. Assumptions and notation

We begin by stating some well-known definitions and results, which show that the classical ruin problem over an infinite time period is a special case of a barrier crossing problem for random walks. The abbreviation "a.s" stands for "almost surely." In this paper the **Erlang**(n, λ) distribution is the same as the **Gamma**(n, λ), but the former is restricted to $n \in \mathbb{N}_+$.

By changing the monetary units if necessary, it is always possible to assume that premiums are received at the rate of 1 per time unit. The classical risk process is thus

$$U_t \;=\; u + t - \sum_{j=1}^{N_t} X_j,$$

where $\{N_t\}$ is the counting process generated by i.i.d. (non-negative) waiting times $\{W_j\}$, that is,

$$N_t \;=\; \max\{n \,|\, W_1 + \cdots + W_n \le t\}$$

or 0, if the set on the right is empty. Each of the sequences $\{X_j\}$, $\{W_j\}$ is asumed i.i.d. and non-negative, and all the variables $\{X_j, W_k;\, j, k \ge 1\}$ are assumed independent. Ruin is the event "there is a $t < \infty$ such that $U_t < 0$," or, equivalently,

$$\left\{\omega \;\middle|\; \sup_{0\le t<\infty} \left(\sum_{j=1}^{N_t(\omega)} X_j(\omega) - t \right) \;>\; u \right\}.$$

If $T_n = W_1 + \cdots + W_n$, $T_0 = 0$, then

$$U_{T_n} \;=\; u + \sum_{j=1}^{n} (W_j - X_j),$$

and ruin becomes equivalently

$$\left\{ \sup_{0\le t<\infty} \left(\sum_{j=1}^{N_t} X_j - t \right) \;>\; u \right\} \;=\; \left\{ \inf_{0\le n<\infty} U_{T_n} < 0 \right\}.$$

Let us rephrase the problem as follows. Suppose $\{Y_j\}$ is an arbitrary i.i.d. sequence, and define

$$S_0 \;=\; 0, \qquad S_n \;=\; \sum_{j=1}^{n} Y_j, \qquad n \ge 1.$$

The probability of non-ruin, or probability of survival, becomes

$$\varphi(u) \;=\; \mathsf{P}\,(M \ge -u) \qquad \text{where} \qquad M = \inf_{0\le n<\infty} S_n.$$

Note that $\varphi(u) = 0$ for all $u < 0$. In the context of Risk Theory, we write

$$Y_j \;=\; W_j - X_j,$$

where $\{X_j\}$ represent the claims, and $\{W_j\}$ the waiting times between claims. We denote Y a random variable with the same distribution as Y_j, and similarly for W and X. From the Law of Large Numbers, if $\mathsf{E}Y > 0$ then U_t tends to $+\infty$ a.s. as $t \to \infty$ (this includes the case where the average waiting time has infinite expectation while the average claim is finite). If $\mathsf{E}Y \le 0$, then ruin is certain. In the sequel we always assume that

$$0 \;<\; \mathsf{E}Y \;<\; \infty.$$

A consequence of this assumption is that the variable M defined above is a.s. finite, whence

$$\lim_{u\to\infty} \varphi(u) \;=\; 1.$$

For a random variable V, let F_V denote the measure on $B(\mathbb{R})$ generated by its distribution function. By conditioning on Y_1, we find

$$\varphi(u) \;=\; \int dF_Y(y)\varphi(u+y) \;=\; \mathsf{E}\varphi(u+Y). \tag{2.1}$$

In the case of the risk process described above, Y is the difference of two independent variables W and X, and (2.1) becomes

$$\varphi(u) \;=\; \mathsf{E}\varphi(u+W-X) \;=\; \int dF_W(t) \int dF_X(v)\,\varphi(u+t-v). \tag{2.2}$$

For a complex number s, we use the notations

$$\tilde{\varphi}(s) \;=\; \int_0^\infty du\, e^{-su}\varphi(u),\;\; \tilde{\mathbf{w}}(s) \;=\; \mathsf{E}e^{-sW},\;\; \tilde{\mathbf{x}}(s) \;=\; \mathsf{E}e^{-sX},\;\; \tilde{\mathbf{y}}(s) \;=\; \mathsf{E}e^{-sY} \tag{2.3}$$

whenever these integrals exist, and we use the same symbol for their analytic continuations. The abscissa of holomorphy h_V of a random variable V is defined as

$$\mathsf{h}_V \;=\; \inf\{s\in\mathbb{R} \,|\, \mathsf{E}\, e^{-sV} < \infty\}.$$

For a distribution F_V with $\mathsf{h}_V > -\infty$, the function $s \mapsto \mathsf{E}\, e^{-sV}$ must have a singularity at h_V (Widder, 1946, p.58). For instance, a finite combination of $\mathbf{Erlang}(n_j, \lambda_j)$ distributions has

$$\mathsf{h}_V \;=\; -\min_j\{\lambda_j\}.$$

The positive part of a is denoted $a^+ = \max(a,0)$.

Lemma 2.1. *The transform $\tilde{\varphi}(s)$ in (2.3) is analytic in $\{s \in \mathbb{C} \,|\, \mathsf{Re}(s) > 0\}$. The same applies to $\tilde{\mathbf{w}}(s)$ and $\tilde{\mathbf{x}}(s)$, when $W, X \ge 0$. If $\mathsf{E}Y > 0$, then*

$$\lim_{s\to 0+} s\tilde{\varphi}(s) \;=\; \varphi(\infty) \;=\; 1$$
$$\lim_{s\to \infty} s\tilde{\varphi}(s) \;=\; \varphi(0).$$

Proof. The limits are the usual Initial and Final Value Theorems for Laplace transforms, see Doetsch (1974). The second one applies because $\varphi(u) \;=\; \mathsf{P}(-M \le u)$, and thus $\lim_{u\to 0+} \varphi(u) = \varphi(0)$. □

3. First properties of $\tilde{\varphi}(\mathbf{s})$

The factorization given in Theorem 3.1 is not the one commonly used in the study of random walks (Feller, 1971; Borovkov, 1976). Theorem 3.5 is a lower bound for $\psi(0)$, which the author has not seen in the literature. We assume $W, X \geq 0$. Since W and X are independent, $Y = W - X$ implies $\tilde{\mathbf{y}}(s) = \tilde{\mathbf{w}}(s)\tilde{\mathbf{x}}(-s)$, and thus $\mathsf{h}_{Y+} = \mathsf{h}_W$. The restriction $\mathsf{h}_W < 0$ may be removed in Theorems 3.2-3.5, see Section 7.

Theorem 3.1. *Suppose* $\mathsf{h}_W < 0$. *Then, for* $0 < \mathsf{Re}(s) < -\mathsf{h}_W$,

$$\tilde{\varphi}(s) \;=\; \frac{n(s)}{d(s)}, \tag{3.1}$$

where

$$n(s) \;=\; \mathsf{E}\int_0^\infty dv\, e^{sv}\varphi(Y - v) \;=\; \mathsf{E}\int_0^{Y^+} dv\, e^{sv}\varphi(Y^+ - v) \tag{3.2}$$

$$\;=\; \mathsf{E}\int_0^{W} dv\, e^{s(W-v)}\varphi(v - X) \tag{3.3}$$

$$d(s) \;=\; \tilde{\mathbf{y}}(-s) - 1. \tag{3.4}$$

Proof. Multiplying (2.1) by e^{-su} and integrating yields (for $0 < \mathsf{Re}(s) < -\mathsf{h}_W$)

$$\begin{aligned}
\tilde{\varphi}(s) &= \mathsf{E}\, e^{sY}\int_0^\infty du\, e^{-s(u+Y)}\varphi(u+Y) \\
&= \mathsf{E}\, e^{sY}\int_{Y^+}^\infty dv\, e^{-sv}\varphi(v) \\
&= \mathsf{E}\, e^{sY}\left[\int_0^\infty dv\, e^{-sv}\varphi(v) - \int_0^{Y^+} dv\, e^{-sv}\varphi(v)\right] \\
&= \tilde{\mathbf{y}}(-s)\tilde{\varphi}(s) - \mathsf{E}\int_0^{Y^+} du\, e^{su}\varphi(Y^+ - u).
\end{aligned}$$

Since

$$\left|\int_0^{Y^+} dv\, e^{sv}\varphi(Y^+ - v)\right| \;\leq\; Y^+ e^{\mathsf{Re}(s)Y^+}, \tag{3.5}$$

the second term in the last expression is finite if $0 \leq \mathsf{Re}(s) < -\mathsf{h}_W$. This proves (3.1), (3.2) and (3.4). From (2.2) (if $0 < \mathsf{Re}(s) < -\mathsf{h}_W$),

$$\begin{aligned}
\tilde{\varphi}(s) &= \mathsf{E}\, e^{sW}\int_0^\infty du\, e^{-s(u+W)}\varphi(W - X + u) \\
&= \mathsf{E}\, e^{sW}\int_W^\infty dv\, e^{-sv}\varphi(v - X) \\
&= \mathsf{E}\, e^{sW}\int_0^\infty dv\, e^{-sv}\varphi(v - X) - \mathsf{E}\int_0^{W} dv\, e^{s(W-v)}\varphi(v - X).
\end{aligned}$$

The independence of W and X implies that the first term in the last expression equals

$$\tilde{\mathbf{w}}(-s)\int_0^\infty dv\, e^{-sv}\int_0^v dF_X(x)\varphi(v-x) \;=\; \tilde{\mathbf{w}}(-s)\tilde{\mathbf{x}}(s)\tilde{\varphi}(s),$$

which proves (3.3). □

Theorem 3.2. *Suppose* $\mathsf{h}_W < 0$. *Then the function* $n(s)$ *of Theorem 3.1 is analytic in* $\{s \in \mathbb{C} \,|\, \mathsf{Re}(s) < -\mathsf{h}_W\}$, *and*

$$|n(s)| \;\le\; n(0) \;=\; \mathsf{E}\int_0^{Y^+} \varphi(Y^+ - u)\, du \;\le\; \mathsf{E}(Y^+), \qquad \mathsf{Re}(s) \le 0.$$

Suppose $n(s)$ *and* $d(s)$ *have analytic continuations in a domain* D *of* $\{\mathsf{Re}(s) > 0\}$; *if* $s_0 \in D$ *is a zero of* $d(s)$, *then it is also a zero of* $n(s)$, *with at least the same multiplicity.*

Proof. Recall (3.5), and note that $\tilde{\varphi}(s)$ has no pole in $\{s \in \mathbb{C} \,|\, \mathsf{Re}(s) > 0\}$. □

Theorem 3.3. *Suppose* $\mathsf{h}_W < 0$. *If* $\mathsf{E}\, e^{-sY^-}$ *(or* $\tilde{\mathbf{x}}(s)$*) has an analytic continuation in some domain* D *of* $\{\mathsf{Re}(s) \le 0\}$, *then* $\tilde{\varphi}(s)$ *has an analytic continuation in* D *as well, where it is equal to* $n(s)/d(s)$.

Proof. We know that $n(s)$ is analytic to the left of $-\mathsf{h}_W$ (Theorem 3.2); moreover, $d(s) = \tilde{\mathbf{w}}(-s)\tilde{\mathbf{x}}(s) - 1$ is analytic in the strip $\{0 < \mathsf{Re}(s) < -\mathsf{h}_W\}$. Therefore, if $\tilde{\mathbf{x}}(s)$ has an analytic continuation to the left of that strip, then we must have $\tilde{\varphi}(s) = n(s)/d(s)$ there too. Next, observe that

$$\begin{aligned}\tilde{\mathbf{y}}(-s) \;=\; \mathsf{E}\, e^{sY} \;&=\; \mathsf{E}\, e^{sY}\mathbf{1}_{\{Y\le 0\}} + \mathsf{E}\, e^{sY}\mathbf{1}_{\{Y>0\}}\\ &=\; \mathsf{E}\, e^{-sY^-} + \mathsf{E}\, e^{sY^+} - 1.\end{aligned}$$

The function $\mathsf{E}\, e^{-sY^-}$ is analytic in the right half-plane, while $\mathsf{E}\, e^{sY^+}$ is analytic to the left of h_W. Consequently, if $\mathsf{E}\, e^{-sY^-}$ has an analytic continuation in a subset of the left half-plane, so does $\tilde{\mathbf{y}}(-s)$. □

Theorem 3.4. *If* $\mathsf{h}_W < 0$, *then*

$$n(0) \;=\; \mathsf{E}\int_0^{Y^+} \varphi(Y^+ - u)\, du \;=\; \mathsf{E}(Y).$$

Proof. The result follows from

$$1 \;=\; \varphi(\infty) \;=\; \lim_{s\to 0+} s\tilde{\varphi}(s) \;=\; \frac{n(0)}{\lim_{s\to 0+}\frac{d(s)}{s}} \;=\; \frac{n(0)}{\mathsf{E}\, Y}. \qquad \square$$

Theorem 3.5. *If* $\mathsf{h}_W < 0$*, then* $\psi(0) \geq \dfrac{\mathsf{E}\,Y^-}{\mathsf{E}\,Y^+}$.

Proof. From Theorem 3.4,

$$\mathsf{E}\,Y^+ - \mathsf{E}\,Y^- = \mathsf{E}\,Y = \mathsf{E}\int_0^{Y^+} \varphi(Y^+ - u)\,du \geq \varphi(0)\mathsf{E}\,Y^+ = \mathsf{E}\,Y^+ - \psi(0)\mathsf{E}\,Y^+. \;\square$$

Example 3.6. This example takes advantage of the simplifications which occur when Y has an arithmethic distribution over some mutiple of the integers $1, 0, -1, \ldots$ (we let that multiple be one for simplicity), and shows that the inequality in Theorem 3.5 cannot be improved upon in general. Since S_n can only decrease by integer values, $\varphi(u)$ is constant between successive integers; $\varphi(u)$ is also right-continuous. Because Y^+ equals only 0 or 1, we find

$$\mathsf{E}\,Y^+ - \mathsf{E}\,Y^- = \mathsf{E}\int_0^{Y^+} \varphi(Y^+ - u)\,du = \varphi(0)\mathsf{E}\,Y^+.$$

Thus,

$$\psi(0) = \frac{\mathsf{E}\,Y^-}{\mathsf{E}\,Y^+} = \frac{\sum_{k=1}^{\infty} k\mathsf{P}(Y=-k)}{\mathsf{P}(Y=1)}.$$

In particular, if $1/2 < p < 1$, $q = 1-p$, $Y = 1$ with probability p and $Y = -1$ with probability q, then the above implies

$$\psi(0) = \frac{q}{p},$$

which can be checked in other ways (for instance, apply the theorem on p.413 of Bowers *et al.* (1997).) $\square$

Removing zero claims or waiting times

In the classical Poisson case (and for compound distributions in general), a useful technique is the removal of the zero claim, that is, transforming a given problem so that the probability of a claim being equal to 0 is 0. This is possible more generally for the class of risk models we consider in this paper, for claims as well as for waiting times. For instance, suppose $\mathsf{P}(W=0) = a_0 \in (0,1)$, and let

$$N_0 = 0, \qquad N_k = \min\{n > N_{k-1} \,|\, W_n > 0\}, \qquad k \geq 1.$$

Define

$$\bar{S}_n = \sum_{j=1}^{n} (\bar{W}_j - \bar{X}_j), \qquad \bar{W}_j = W_{N_j}, \qquad \bar{X}_j = \sum_{\ell=N_{j-1}+1}^{N_j} X_j.$$

The distribution of $\bar{X}$ is compound geometric, with probability of succcess $1-a_0$. Since the ruin probability is unaffected by the change from S_n to $\bar{S}_n$, the ruin problem with

$$\begin{aligned} F_{\bar{W}}(x) &= \frac{F_W(x)-a_0}{1-a_0}, \qquad x \geq 0, \qquad \int_0^\infty e^{-st}\,dF_{\bar{W}}(t) = \frac{\tilde{\mathbf{w}}(s)-a_0}{1-a_0} \\ F_{\bar{X}}(x) &= \sum_{k=1}^{\infty} F_X^{*k}(x)(1-a_0)a_0^{k-1}, \qquad \int_0^\infty e^{-st}\,dF_{\bar{X}}(t) = \frac{(1-a_0)\tilde{\mathbf{x}}(s)}{1-a_0\tilde{\mathbf{x}}(s)} \end{aligned} \tag{3.6}$$

is equivalent to the original one, but has no zero waiting time. Removing the zero claim leads to a similar equivalent problem, with the roles of W and X reversed.

4. Probability laws with rational Laplace transforms

A *rational function* is the ratio of two polynomials.

Definition. *A probability distribution μ on $\mathbb{R}$ is said to belong to $\mathcal{R}^f$ if its Laplace transform is a rational function. If μ is concentrated on $\mathbb{R}_+$, then it will be said to belong to $\mathcal{R}^f_+$. In either case the distribution will be said to be* ***rational****.*

The class $\mathcal{R}^f_+$ includes all combinations of Erlang densities, possibly including a mass at the origin. For instance, the measure μ which is a combination of a mass $a_0 < 1$ at the origin, and $1-a_0$ times an **Erlang**(n,λ) density, n a positive integer, has Laplace transform

$$a_0 + (1-a_0)\left(\frac{\lambda}{\lambda+s}\right)^n = \frac{a_0(\lambda+s)^n + (1-a_0)\lambda^n}{(\lambda+s)^n} = \frac{P_1(s)}{P_2(s)}.$$

where P_1, P_2 are polynomials. When the degrees of P_1 and P_2 are the same, as they are above when $a_0 > 0$, then necessarily there is a point mass at the origin. If there is no mass at the origin, then the degree of P_1 is lower than the degree of P_2. With the help of Eqs.(3.6), any mass at the origin in the distribution of $W \in \mathcal{R}^f_+$ may be removed, and the transformed distribution is still in $\mathcal{R}^f_+$; the same applies to the removal of zero claims.

The combinations of Erlang densities do not necessarily have only positive weights; for instance, the function

$$(3e^{-2x} - 2e^{-4x})\mathbf{1}_{\{x>0\}}$$

is non-negative and integrates to one, even though it includes the exponential density with parameter 4 multiplied by $-1/2$. All the measures in $\mathcal{R}^f_+$ may be expressed as

$$d\mu(t) = a_0\delta(dt) + \left[\sum_{j=1}^{n}\sum_{k=1}^{d_j} a_{jk}\frac{b_j^{c_{jk}}\,t^{c_{jk}-1}\,e^{-b_j t}}{(c_{jk}-1)!}\mathbf{1}_{(0,\infty)}(t)\right]dt, \tag{4.1}$$

where

$$a_0 + \sum_{j,k} a_{jk} \;=\; 1.$$

Here, the $\{b_j\}$ are either real and positive, or else complex numbers with positive real parts, and the $\{c_{jk}\}$ are positive integers; for convenience we further assume $c_{jm} \le c_{jd_j}$ for all $m \le d_j$ and $a_{jd_j} \ne 0$. The Laplace transform of F may be expressed as P_1/P_2 where P_1, P_2 are irreducible polynomials, with

$$P_2(s) \;=\; (b_1+s)^{c_{1d_1}} \cdots (b_n+s)^{c_{nd_n}}.$$

The $\{b_j\}$ are then the poles of the Laplace transform of F. When the $\{b_j\}$ are all real, μ is seen to be a combination of a mass at the origin and of **Erlang**(c_{jk}, b_j) densities, for $1 \le k \le d_j$, $1 \le j \le n$. In other words, the density of the continuous part of μ is of the from

$$\sum_{j=1}^{n} \pi_j(x) e^{-b_j x}, \tag{4.2}$$

where the $\pi_j(x)$ is a polynomial in x. The roots of P_2, $b_1, \ldots, b_n$, are the scale parameters of the Erlang densities which make up the combination. Observe that b_j has multiplicity c_{jd_j}.

Some of the parameters $\{b_j\}$ may be complex, so that the class $\mathcal{R}_+^f$ contains, in particular, measures which have damped sine or cosine functions as part of their densities. For example, the measure with density

$$\frac{17}{13} e^{-x} \left[1 - \sin(4x)\right] \mathbf{1}_{\{x>0\}} \tag{4.3}$$

is non-negative, integrates to one, and has Laplace transform

$$\frac{17}{13}\left(\frac{1}{1+s} - \frac{4}{(s+1)^2+16}\right) \;=\; \frac{17}{13} \cdot \frac{s^2-2s+13}{(1+s)[(s+1)^2+16]}, \tag{4.4}$$

and thus belongs to $\mathcal{R}_+^f$. Here the three roots of the denominator are -1, $-1+4i$ and $-1-4i$. The last two are complex and conjugate. In all cases where a rational distribution has complex roots, the latter have to come in pairs of conjugate roots (otherwise the density would not be a real function). In all other respects, the densities of the distributions in $\mathcal{R}_+^f$ with some complex b_j are just the same as in (4.2). In effect, this means that the class of combinations of Erlang densities naturally includes damped sines and cosines, although such functions cannot individually be probability densities. Note that $\min\{\mathsf{Re}(b_j)\}$ has to be one of the real $\{b_j\}$, and that the abscissa of holmorphy of μ is $\mathsf{h}_\mu = -\min\{\mathsf{Re}(b_j)\}$.

The class $\mathcal{R}_+^f$ strictly contains the class of phase type distributions since the latter does not allow damped trigonometric functions as components, see Neuts (1981, Chapter 2). An early reference to rational distributions is Cox (1955). One might initially believe that ruin problems would be simpler if complex $\{b_j\}$ are excluded in the distributions of waiting times or claims, but the analysis below shows that it is not so. The formulas are the same, and the tools used to derive them are identical. Even when all the $\{b_j\}$ are real, the probability of ruin may still contain damped trigonometric functions, as can be seen in Dickson (1998, Section 6) and in Section 6 below.

5. Waiting times in $\mathcal{R}_+^f$

This section shows how to obtain the Laplace transform of the probabiity of survival, $\tilde{\varphi}(s)$, when the distribution of waiting times is in $\mathcal{R}_+^f$. This is achieved by showing that $n(s)$ is a rational function, which can be determined explicitly. In general, this does not yield the probability of ruin, except in specific cases. The situation is the same as in the classical Poisson case: inverting the Laplace transform for φ depends on the Laplace transform of the distribution for the claims, and may be a difficult problem; the inversion of $\tilde{\varphi}(s)$ can be performed explicitly in cases where the claims are in $\mathcal{R}^f$, as shown in Section 6.

Theorem 5.1. *Let $\{b_j\}$ be complex numbers with positive real part, and $\{c_{jk}, d_j\}$ positive integers, with $c_{jm} \le c_{jd_j}$ for all $m \le d_j$, $a_{jd_j} \ne 0$, and suppose the distribution of W is given by (4.1). Then*

$$n(s) \;=\; \sum_{j=1}^{n}\sum_{k=1}^{d_j}\sum_{m=0}^{c_{jk}-1} \frac{a_{jk} b_j^{c_{jk}} (-1)^m (\tilde{\mathbf{x}}\tilde{\varphi})^{(m)}(b_j)}{m!(b_j - s)^{c_{jk}-m}}$$

for $s \in \mathbb{C} - \{b_1, \ldots, b_n\}$, with $(\tilde{\mathbf{x}}\tilde{\varphi})^{(m)}(s) = (d^m/ds^m)(\tilde{\mathbf{x}}\tilde{\varphi}(s))$.

Proof. Apply (3.3). For $\mathsf{Re}(s) < -\mathsf{h}_W = \min\{\mathsf{Re}(b_j)\}$ and $v > 0$,

$$\mathsf{E}\, e^{sW}\mathbf{1}_{\{W>v\}} \;=\; \sum_{j=1}^{n}\sum_{k=1}^{d_j} \frac{a_{jk} b_j^{c_{jk}}}{(c_{jk}-1)!(b_j - s)^{c_{jk}}} \int_{v(b_j-s)}^{\infty} du\, u^{c_{jk}-1} e^{-u}.$$

Recall the incomplete gamma function (Abramowitz & Stegun, 1972, pp.260-262):

$$\Gamma(a,x) \;=\; \int_x^\infty t^{a-1} e^{-t}\, dt, \qquad x \ge 0.$$

For $c = 1, 2, \ldots$,

$$\Gamma(c,x) \;=\; \Gamma(c)\mathcal{E}(c-1,x)e^{-x} \qquad \text{where} \qquad \mathcal{E}(c-1,x) \;=\; \sum_{m=0}^{c-1} \frac{x^m}{m!}.$$

Thus

$$\mathsf{E}\, e^{sW}\mathbf{1}_{\{W>v\}} \;=\; \sum_{j=1}^{n}\sum_{k=1}^{d_j} \frac{a_{jk} b_j^{c_{jk}}}{(b_j - s)^{c_{jk}}} \mathcal{E}(c_{jk}-1, v(b_j - s)) e^{-v(b_j - s)}$$

and

$$\begin{aligned}\int_0^\infty dv\, e^{-sv}\mathsf{E}\varphi(v-X)\,[v(b_j-s)]^m e^{-v(b_j-s)}\\ &= (b_j-s)^m \int_0^\infty dv\, e^{-vb_j} v^m \int_0^v dF_X(x)\varphi(v-x)\\ &= (b_j-s)^m(-1)^m \frac{d^m}{dr^m}\int_0^\infty dv\, e^{-vr}\int_0^v dF_X(x)\varphi(v-x)\Big|_{r=b_j}\\ &= (b_j-s)^m(-1)^m \frac{d^m}{dr^m}[\tilde{\mathbf{x}}(r)\tilde{\varphi}(r)]\Big|_{r=b_j}.\end{aligned}$$

The calculation was done under the assumption that $\mathsf{Re}(s) < -\mathsf{h}_W$, but $d(s) = \tilde{\mathbf{w}}(-s)\tilde{\mathbf{x}}(s) - 1$ is analytic, except for poles, in $\{\mathsf{Re}(s) > 0\}$, and so must $n(s) = \tilde{\varphi}(s)d(s)$. □

Since $\tilde{\varphi}(s)$ has no poles in the right half-plane, the unknown constants $(\tilde{\mathbf{x}}\tilde{\varphi})^{(m)}(b_j)$ in $n(s)$ may be found by determining the zeros of the denominator $\tilde{\mathbf{y}}(-s) - 1$ in that half-plane.

Theorem. (Rouché) *Suppose $f(z)$ and $g(z)$ are analytical on and within a closed contour Γ in $\mathbb{C}$. Suppose (a) $f(z)$ does not vanish on Γ , and (b) $|g(z)| < |f(z)|$ on Γ. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros within Γ.*

Theorem 5.2. *Suppose $\mathsf{h}_X < 0$ and that W is the rational distribution of Theorem 5.1. Let*

$$\pi(s) \;=\; \prod_{j=1}^{n}(b_j-s)^{c_{jd_j}}.$$

Then the polynomial

$$N(s) \;=\; \pi(s)n(s),$$

has degree $\nu = c_{1d_1} + \cdots + c_{nd_n} - 1$, and the function $D(s) = \pi(s)d(s)$ is analytic in $\{\mathsf{Re}(s) > 0\}$. Moreover, there is $R > \max_j |b_j|$ such that

$$|w(-s)| < 1 \quad \forall\, s \text{ with } |s| = R,\ \mathsf{Re}(s) \geq 0. \tag{5.1}$$

Let C_R be the closed path consisting of the half-circle $\{s\big|\,|s| = R, \mathsf{Re}(s) > 0\}$ and the line segment going from $-R\,i$ to $+R\,i$. Then $N(s)$ and $D(s)$ have exactly ν zeros inside C_R. Moreover,

$$N(s) \;= \sum_{j=0}^{\nu} p_j s^j \quad \text{with} \quad p_0 = \mathsf{E}Y\prod_{j=1}^{n} b_j^{c_{jd_j}} \quad \text{and} \quad p_\nu = \varphi(0)(-1)^\nu.$$

Proof. By Theorem 5.1, $N(s)$ is a polynomial, with degree no larger than ν. Furthermore, $D(s) = \pi(s)d(s)$ is analytic in $\{\mathsf{Re}(s) > 0\}$, because multiplying by $(b_j - s)^{c_j d_j}$ cancels the pole of $\tilde{\mathbf{w}}(-s)$ at b_j, and $\tilde{\mathbf{x}}(s)$ is of course analytic in that region. We can write the Laplace transform of W as

$$a_0 + \frac{P_1(s)}{P_2(s)},$$

where the degree of P_1 is strictly smaller than the degree of P_2; this implies that $\tilde{\mathbf{w}}(-s)$ tends to $a_0 < 1$ as $|s|$ tends to ∞, and that (5.1) is satisfied for some $R < \infty$. Choose $\mathsf{h}_X < -q < 0$ such that $\tilde{\mathbf{y}}(-q) < 1$ (this is possible because $\mathsf{E}\, Y > 0$) and form the closed path $C_{R,q}$ (see Figure 1) consisting of the half-circle $\{s \,|\, \mathsf{Re}(s) = R,\ \mathsf{Re}(s) > 0\}$, the line segment from Ri to $-q$, and finally the line segment from $-q$ to $-Ri$. On the line segments we have $s = u + iv$, $u < 0$, $v \in \mathbb{R}$, and so

$$|\tilde{\mathbf{y}}(u + iv)| \ \leq\ \tilde{\mathbf{y}}(u) \ <\ 1.$$

We also have $|\tilde{\mathbf{y}}(\pm iR)| < 1$, since the distribution of Y is not arithmetic (Feller, 1971, p.501). Thus $|\tilde{\mathbf{y}}(-s)| < 1$ for all $s \in C_{R,q}$, so

$$|\pi(s)\tilde{\mathbf{y}}(-s)| \ <\ |\pi(s)|, \quad s \in C_{R,q},$$

as $\pi(s)$ does not vanish on $C_{R,q}$. From Rouché's Theorem, $\pi(s)\tilde{\mathbf{y}}(-s) - \pi(s)$ has the same number of zeros inside $C_{R,q}$ as $\pi(s)$, that is, $\nu + 1$. Let q tend to zero. Observe that $d(s)$ has a zero at the origin, which must have multiplicity one, for $\tilde{\mathbf{y}}'(0) = \mathsf{E}\, Y \neq 0$. Then the number of zeros of $\pi(s)d(s)$ inside C_R is ν.

By Theorem 3.4,

$$p_0 \ =\ N(0) \ =\ \pi(0)n(0) \ =\ \pi(0)\mathsf{E}\, Y.$$

Finally, by Lemma 2.1,

$$\begin{aligned} p_\nu &= \lim_{s\to\infty} \frac{N(s)}{s^\nu} \ =\ \lim_{s\to\infty} s\tilde{\varphi}(s)\frac{\pi(s)[\tilde{\mathbf{y}}(-s) - 1]}{s^{\nu+1}} \\ &= \varphi(0) \lim_{s\to\infty} \frac{\pi(s)}{s^{\nu+1}} \lim_{s\to\infty} [\tilde{\mathbf{w}}(-s)\tilde{\mathbf{x}}(s) - 1] \\ &= -\varphi(0) \lim_{s\to\infty} \prod_{j=1}^{n} \left(\frac{b_j}{s} - 1\right)^{c_j d_j} \ =\ -\varphi(0)(-1)^{\nu+1}. \end{aligned}$$

□

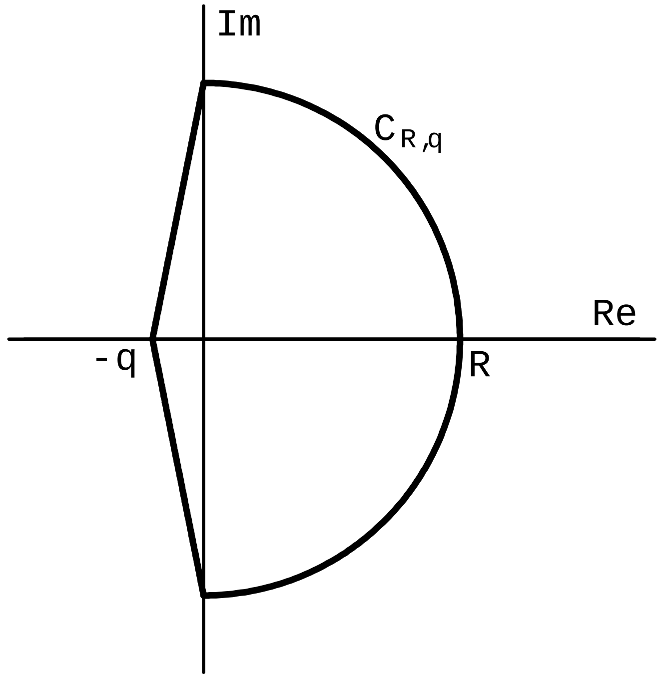

Figure 1. The path $C_{R,q}$ in the complex plane

Example 5.3. Suppose $W \sim \mathbf{Erlang}(\alpha, \lambda)$, that is,

$$dF_W(t) \;=\; \frac{\lambda^\alpha t^{\alpha-1} e^{-\lambda t}}{(\alpha-1)!}\mathbf{1}_{(0,\infty)}(t)dt.$$

Then

$$n(s) \;=\; \sum_{m=0}^{\alpha-1} \frac{(-1)^m \lambda^\alpha (\tilde{\mathbf{x}}\tilde{\varphi})^{(m)}(\lambda)}{m!(\lambda-s)^{\alpha-m}}, \qquad \pi(s) \;=\; (\lambda-s)^\alpha,$$

and $N(s)$ is a polynomial of degree $\nu = \alpha - 1$. From Theorem 5.2 ,

$$p_0 \;=\; \lambda^\alpha \mathsf{E}\, Y, \qquad p_{\alpha-1} \;=\; \varphi(0)(-1)^{\alpha-1}.$$

The roots of $D(s) = (\lambda - s)^\alpha[\tilde{\mathbf{w}}(-s)\tilde{\mathbf{x}}(s) - 1]$ must all satisfy

$$\tilde{\mathbf{x}}(s) \;=\; 1/\tilde{\mathbf{w}}(-s) \;=\; \left(1 - \frac{s}{\lambda}\right)^\alpha. \tag{5.2}$$

The resulting expression for $\tilde{\varphi}(s)$ agrees with the one given by Kalashnikov (1998, Eq.(1)), which is itself taken from Prabhu (1980), since

$$\frac{N(s)}{D(s)} \;=\; \frac{\sum_{j=0}^{\alpha-1} p_j s^j}{\pi(s)d(s)}$$

$$= \frac{p_0 \prod_{j=0}^{\alpha-1}(1-\frac{s}{\rho_j})}{(\lambda-s)^\alpha[w(-s)x(s)-1]}$$

$$= \frac{\lambda^\alpha \mathsf{E}(W-X)\prod_{j=0}^{\alpha-1}(1-\frac{s}{\rho_j})}{\lambda^\alpha x(s)-(\lambda-s)^\alpha}$$

$$= \frac{(\alpha-\lambda\mathsf{E}X)\prod_{j=0}^{\alpha-1}(\lambda-\frac{\lambda s}{\rho_j})}{\lambda^\alpha x(s)-(\lambda-s)^\alpha},$$

if $\{\rho_j\}$ are the non-zero roots of $d(s)$.

Consider the real roots of Eq.(5.2). Because $\tilde{\mathbf{y}}(-s) = \mathsf{E}\exp(sY)$ is convex, with

$$\frac{d}{ds}\tilde{\mathbf{y}}(-s)\bigg|_{s=0} = \mathsf{E}Y > 0,$$

$D(s)$ cannot have a real root between 0 and λ. Suppose α is even. For $s \geq \lambda$, $\tilde{\mathbf{x}}(s)$ is decreasing, $1/\tilde{\mathbf{w}}(-s)$ is increasing, and so $D(s)$ has just one real, positive root, larger than λ (see Figure 2). If α is odd, there is no real root (Figure 3), since $1/\tilde{\mathbf{w}}(-s)$ is negative for $s > \lambda$. In short, *$N(s)$ has at most one real root, and, if it has one, it is larger than λ. All other roots of $N(s)$ are complex and conjugate.* Finally, if $\mathsf{Re}(s) > 0$ and $|\lambda - s| \geq \lambda$, then $1/\tilde{\mathbf{w}}(-s) \geq 1$ and $x(s) < 1$, so that s cannot be a solution of (5.2), and thus *all roots $\{\rho_j\}$ are within the circle of radius λ centered at $s = \lambda$.*

Let us consider the cases $\alpha = 1, 2$ in more detail. The case $\alpha = 1$ is the classical risk model, with claims arriving according to a homogeneous Poisson process with intensity λ. Let us derive the formula for the Laplace transform of the derivative of the probability of ruin, which is given in Bowers *et al.*, p.419:

$$\int_0^\infty e^{-su}[-\psi'(u)]\,du = \frac{\theta}{1+\theta}\cdot\frac{1-\tilde{\mathbf{x}}(s)}{\tilde{\mathbf{x}}(s)+(1+\theta)s\mathsf{E}X-1}. \tag{5.3}$$

In our notation, the security loading is

$$\theta = \frac{\mathsf{E}Y}{\mathsf{E}X}, \tag{5.4}$$

and so (5.3) becomes

$$\lambda\mathsf{E}Y\cdot\frac{1-\tilde{\mathbf{x}}(s)}{\tilde{\mathbf{x}}(s)+\frac{s}{\lambda}-1}.$$

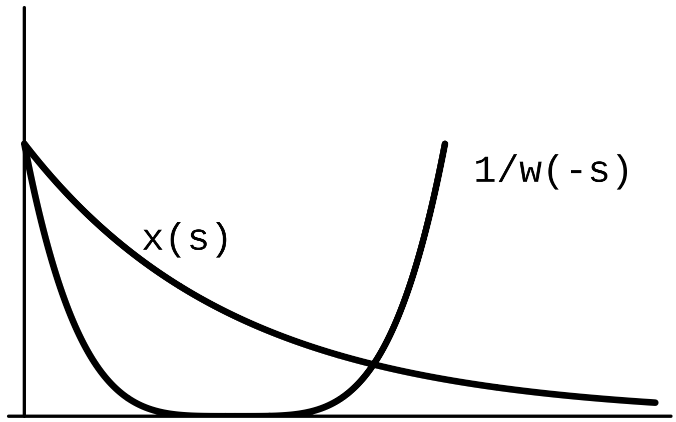

Figure 2. $\tilde{\mathbf{x}}(s)$ and $1/\tilde{\mathbf{w}}(-s)$, n even

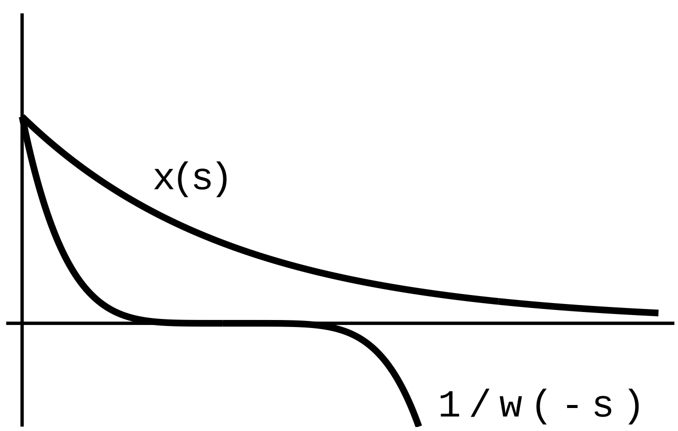

Figure 3. $\tilde{\mathbf{x}}(s)$ and $1/\tilde{\mathbf{w}}(-s)$, n odd

Now, by Theorem 5.2, $N(s)$ has degree 0, and

$$p_0 \;=\; \lambda \mathsf{E} Y \;=\; \varphi(0).$$

Hence

$$\begin{aligned}\int_0^\infty e^{-su}[-\psi'(u)]\,du &= \int_0^\infty e^{-su}\varphi'(u)\,du \\ &= -\varphi(0) + s\tilde{\varphi}(s) \\ &= -\lambda \mathsf{E} Y + \frac{s\lambda \mathsf{E} Y}{\lambda\tilde{\mathbf{x}}(s) - \lambda + s} \\ &= \lambda \mathsf{E} Y \cdot \frac{1-\tilde{\mathbf{x}}(s)}{\tilde{\mathbf{x}}(s) + \frac{s}{\lambda} - 1}.\end{aligned}$$

When $\alpha = 2$, $N(s)$ has degree 1, and

$$\tilde{\varphi}(s) \;=\; \frac{N(s)}{D(s)} \;=\; \frac{p_0 + p_1 s}{(\lambda - s)^2\left[\frac{\lambda^2}{(\lambda-s)^2}\tilde{\mathbf{x}}(s) - 1\right]} \;=\; \frac{p_0 + p_1 s}{\lambda^2\tilde{\mathbf{x}}(s) - (\lambda - s)^2}.$$

From Theorem 5.2, $p_0 = \lambda^2 \mathsf{E} Y = 2\lambda - \lambda^2 \mathsf{E} X$, and $p_1 = -\varphi(0)$. The latter may be found by determining the root of $D(s)$ in the interval (λ, ∞), say ρ. We must have $N(\rho) = 0$, so $p_1 = -p_0/\rho = -\lambda^2 \mathsf{E} Y/\rho$, from which

$$\varphi(0) \;=\; \frac{\lambda^2 \mathsf{E} Y}{\rho}.$$

Let us be check that $\varphi(0)$ is strictly smaller than one. For s real, the two curves $1/\tilde{\mathbf{w}}(-s)$ and $\tilde{\mathbf{x}}(s)$ meet at the origin, where they have slopes $\tilde{\mathbf{x}}'(0) = -\mathsf{E}\, X$ and $\tilde{\mathbf{w}}'(0)/\tilde{\mathbf{w}}(0)^2 = -\mathsf{E}\, W = -2/\lambda$, respectively. Since $\mathsf{E} W > \mathsf{E} X$, the curve $\tilde{\mathbf{x}}(s)$ is above the curve $1/\tilde{\mathbf{w}}(-s)$ for small positive s. The curves also meet a $\rho > 0$. Because $\tilde{\mathbf{x}}(\cdot)$ is convex, ρ has to be to the right of the point where the tangent of $\tilde{\mathbf{x}}(r)$ at $r = 0$ meets $1/\tilde{\mathbf{w}}(-s)$. That point s_0 satisfies

$$1 - s_0 \mathsf{E} X \;=\; \left(\frac{\lambda - s_0}{\lambda}\right)^2,$$

which means

$$s_0 \;=\; \lambda^2\left(\frac{2}{\lambda} - \mathsf{E} X\right) = \lambda^2 \mathsf{E} Y \;<\; \rho. \qquad \square$$

6. Claims in $\mathcal{R}_+^f$

Remarkable simplifications occur when we assume that $X \in \mathcal{R}_+^f$. Suppose $\mathsf{h}_W < 0$ and that the distribution of X is given by (4.1). We know that $\tilde{\varphi}(s)$ is analytic in $\{\mathsf{Re}(s) > 0\}$, and that $n(s)$ is analytic in $\{\mathsf{Re}(s) < -\mathsf{h}_W\}$ (Theorem 3.2). Moreover, $d(s) = \tilde{\mathbf{w}}(-s)\tilde{\mathbf{x}}(s) - 1$ is analytic in $\{\mathsf{Re}(s) < -\mathsf{h}_W\}$, except for a finite number of poles in $\{\mathsf{Re}(s) < 0\}$. Then $\tilde{\varphi}(s) = n(s)/d(s)$ is analytic in $\mathbb{C}$, except for a finite number of poles. Each of these poles has to be a zero of $d(s)$.

Theorem 6.1. *If $\mathsf{h}_W < 0$ and $X \in \mathcal{R}_+^f$, then $\tilde{\varphi}(s)$ is a rational function. If, moreover, no zero of $\tilde{\mathbf{w}}(-s)$ is a pole of $\tilde{\mathbf{x}}(s)$, then the non-zero roots of $d(s)$ are all in $\{\mathsf{Re}(s) < 0\}$, and their number equals the number of poles of $\tilde{\mathbf{x}}(s)$.*

If the distribution of X is a combination of **Erlang**(m_j, β) *distributions (possibly with a mass at the origin), then no zero of $\tilde{\mathbf{w}}(-s)$ can be a pole of $\tilde{\mathbf{x}}(s)$.*

Proof. With respect to the last claim, a combination of **Erlang**(m_j, β) plus a mass at 0 has a Laplace transform with real poles only; any zero of $\tilde{\mathbf{w}}(-s)$ in $\{\mathsf{Re}(s) \leq 0\}$ has to be complex, since this is the Laplace transform of a real, non-negative measure.

Obviously, $s = 0$ is a root of $d(s)$. Here Y cannot have an arithmetic distribution, so $d(s)$ has no other zero on the imaginary axis (Feller, 1971, p.501). Next, we show how to form a closed contour in the left half plane which contains all the poles of $\tilde{\mathbf{x}}(s)$ in its interior, and on which

$$|\tilde{\mathbf{w}}(-s)| \;<\; \frac{1}{|\tilde{\mathbf{x}}(s)|}. \tag{6.1}$$

Since $\mathsf{E}Y > 0$, and also because $\tilde{\mathbf{y}}(-s)$ exists for s in a neighbourhood of the origin, there is a strip $-\delta < \mathsf{Re}(s) < 0$, $\delta > 0$, where $\tilde{\mathbf{x}}(s)$ has no pole and $|\tilde{\mathbf{y}}(-s)| < 1$. Now, $|\tilde{\mathbf{w}}(-s)| \leq 1$ if $\mathsf{Re}(s) \leq 0$, while $|\tilde{\mathbf{x}}(s)| < 1$ if $|s|$ is large enough, which implies that $d(s)$ cannot vanish if $\mathsf{Re}(s) \leq 0$ and if $|s|$ is larger than some number R_0. Thus, we may choose R such that the half-circle

$$C_R \;=\; \left\{ s = -\frac{\delta}{2} + iy, -R \leq y \leq R; s = -\frac{\delta}{2} + Re^{i\theta}, \frac{\pi}{2} < \theta < \frac{3\pi}{2} \right\}$$

encloses all the poles of $\tilde{\mathbf{x}}(s)$, as well as all the zeros of $d(s)$, and (6.1) holds on the half-circle. We cannot apply Rouché's Theorem to $\tilde{\mathbf{w}}(-s)$ and $1/\tilde{\mathbf{x}}(s)$ based on (6.1), because $1/\tilde{\mathbf{x}}(s)$ is not analytic in C_R. However, recall that

$$\tilde{\mathbf{x}}(s) \;=\; \frac{P_1(s)}{P_2(s)},$$

where $P_1(s), P_2(s)$ are polynomials, expressed in irreducible form. Letting

$$g(s) \;=\; P_1(s)\tilde{\mathbf{w}}(-s) \qquad \text{and} \qquad f(s) \;=\; -P_1(s)/\tilde{\mathbf{x}}(s) \;=\; -P_2(s),$$

we see that the assumptions in Rouché's Theorem are satisfied in C_R, and so

$$h(s) \;=\; f(s) + g(s) \;=\; P_1(s)\tilde{\mathbf{w}}(-s) - P_2(s) \;=\; P_2(s)d(s)$$

has the same number of zeros inside C_R as $P_2(s)$. Every zero of $d(s)$ is a zero of $h(s)$; it remains to show the converse. If $h(s) = 0$ and $P_2(s) \neq 0$, then clearly $d(s) = 0$. If $h(s) = 0$ and $P_2(s) = 0$, then s is one of the poles of $\tilde{\mathbf{x}}$, and so $P_1(s)\tilde{\mathbf{w}}(-s) \neq 0$, a contradiction, since no zero of $\tilde{\mathbf{w}}(-s)$ is a pole of $\tilde{\mathbf{x}}(s)$, and $P_1(s)/P_2(s)$ is irreducible. □

Since $\varphi(u)$ tends to 1 as $u \to \infty$, the poles of $\tilde{\varphi}(s)$ must have negative real parts. There are cases where $\tilde{\mathbf{w}}(-s)$ has zeros in the left half-plane, an example is Eqs.(4.3)-(4.4). These zeros must be complex and conjugate. If such a pair of (simple) zeros $(-b_1, -\bar{b}_1)$ also happen to be (simple) poles of $\tilde{\mathbf{x}}(s)$, then there is a cancellation in the product $\tilde{\mathbf{w}}(-s)\tilde{\mathbf{x}}(s)$, which in effect decreases the number of poles of $\tilde{\mathbf{x}}(s)$ by 2. The proof of the above theorem is unchanged, except that we define

$$\tilde{g}(s) \;=\; \frac{P_1(s)\tilde{\mathbf{w}}(-s)}{(b_1+s)(\bar{b}_1+s)}, \qquad \tilde{f}(s) \;=\; -\frac{P_1(s)/\tilde{\mathbf{x}}(s)}{(b_1+s)(\bar{b}_1+s)} \;=\; -\frac{P_2(s)}{(b_1+s)(\bar{b}_1+s)},$$

and conclude that the number of zeros of $d(s)$ in the left half plane (including the imaginary axis) is one plus the number of poles of $\tilde{\mathbf{x}}(s)$ minus two. The same reasoning shows in general that the number of zeros of $d(s)$ is one plus the number of poles of $\tilde{\mathbf{x}}$ minus the number of zeros of $\tilde{\mathbf{w}}(-s)$ which are also poles of $\tilde{\mathbf{x}}(s)$ (counting multiplicities).

Corollary 6.2. *If* $\mathsf{h}_W < 0$ *and* X *is in* $\mathcal{R}_+^f$*, then*

$$\varphi(u) \;=\; 1 - \sum_{k=1}^{m} f_k(u)e^{-r_k u}, \tag{6.2}$$

where each $f_k(u)$ *is a polynomial, and* $\{-r_k; k = 1, \ldots, m\}$ *are the zeros of* $d(s)$ *in* $\{\mathsf{Re}(s) < 0\}$.

This follows by inverting the rational function $\tilde{\varphi}(s)$. Each pole of $\tilde{\varphi}(s)$ is a zero of $d(s)$ in $\{\mathsf{Re}(s) \leq 0\}$, including 0 (which corresponds to the "1" in the above expression for $\varphi(u)$). The degree of $f_k(u)$ is the multiplicity of r_k minus one, and the sum of the degrees of the $f_k(u)$, plus m, is the number of zeros of $d(s)$ in $\{\mathsf{Re}(s) < 0\}$.

Because $\tilde{\mathbf{x}}(s)$ has a pole at $\mathsf{h}_X = -\min\{\mathsf{Re}(b_j)\}$, and that h_X must be one of the $\{b_j\}$ which are real, we must have

$$\lim_{r \downarrow \mathsf{h}_X} \tilde{\mathbf{y}}(-r) \;=\; +\infty.$$

Now $\tilde{\mathbf{y}}(-r)$ is a convex function for $-r \in (\mathsf{h}_X, 0]$, with $\tilde{\mathbf{y}}'(0) < 0$. Hence $d(-r)$ has exactly one simple, real zero $-r_1$ in the same interval. There cannot be a complex root s of $d(s)$ in $\{-r_1 \le \mathsf{Re}(s) < 0\}$, since we would then have

$$|\mathsf{E}e^{sY}| \;<\; \mathsf{E}e^{\mathsf{Re}(s)Y} \;\le\; 1.$$

(The first strict inequality results because the distribution of Y is not arithmetic.)

In cases where $\tilde{\mathbf{x}}(s)$ has other types of distributions, r_1 may not exist.

Definition. *The smallest $r_1 > 0$ such that $d(-r_1) = \tilde{\mathbf{w}}(r_1)\tilde{\mathbf{x}}(-r_1) = 1$, if it exists, is called the* ***adjustment coefficient***.

In the classical risk model, $\tilde{\mathbf{w}}(r) = \lambda/(\lambda + r)$, and so the relationship which defines the adjustment coefficient is

$$\lambda + r_1 \;=\; \lambda \mathsf{E}e^{r_1 X}$$

or

$$1 + \frac{r_1}{\lambda} \;=\; \mathsf{E}e^{r_1 X}.$$

In our notation, $1/\lambda = (1+\theta)\mathsf{E}X$ (see (5.4)), and thus the above equation is identical to (13.4.3), p.411, in Bowers *et al.* (1997).

Since the adjustment coefficient r_1 is the root of $d(-s)$ with the smallest real part, Corollary 6.2 implies that, when claims are rational,

$$\varphi(u) \;\sim\; Ce^{-r_1 u} \qquad \text{as} \;\; u \to \infty.$$

This is of course a particular case of the famous result due to Cramer (Feller, 1971, p.411).

The hitherto unknown polynomials $\{f_k(\cdot)\}$ may be found by inserting (6.2) into (2.2). We limit ourselves to the simpler case where both $\tilde{\mathbf{x}}(s)$ and $\tilde{\varphi}(s)$ have only simple poles, that is,

$$dF_X(x) \;=\; a_0\delta(dx) + \sum_{j=1}^{m} a_j b_j e^{-b_j x}\mathbf{1}_{(0,\infty)}(x)\,dx, \qquad \varphi(u) \;=\; 1 - \sum_{k=1}^{m} f_k e^{-r_k u},$$

where $\{f_k\}$ are constants. The case where some of the $\{r_k\}$ or $\{b_k\}$ have multiplicities greater than 1 is conceptually the same, but the equations are more complicated (in the case where claims are **Erlang**(α, β), the expression for $\tilde{\varphi}(s)$ is given in Kalashnikov (1998), quoting Prabhu (1980)). Since $b_j \neq r_k$ for all (j, k) (poles cannot be zeros), we get

$$\begin{aligned}
\int dF_X(x)\,\varphi(u+t-x) \;&=\; a_0\varphi(u+t) + \sum_{j=1}^{m} a_j b_j \int_0^{u+t} dx\, e^{-b_j x}\left(1 - \sum_{k=1}^{m} f_k e^{-r_k(u+t-x)}\right) \\
&=\; a_0\varphi(u+t) + \sum_{j=1}^{m} a_j[1 - e^{-b_j(u+t)}] \\
&\quad - \sum_{j=1}^{m}\sum_{k=1}^{m} \frac{a_j b_j f_k}{b_j - r_k}[e^{-r_k(u+t)} - e^{-b_j(u+t)}],
\end{aligned}$$

and so

$$\begin{aligned}\varphi(u) &= 1-\sum_{k=1}^{m} f_k e^{-r_k u} \\ &= a_0 \mathsf{E}\,\varphi(u+W)+\sum_{j=1}^{m} a_j[1-e^{-b_j u}\tilde{\mathbf{w}}(b_j)] \\ &\quad -\sum_{j=1}^{m}\sum_{k=1}^{m}\frac{a_j b_j f_k}{b_j-r_k}[e^{-r_k u}\tilde{\mathbf{w}}(r_k)-e^{-b_j u}\tilde{\mathbf{w}}(b_j)] \\ &= 1-a_0 \mathsf{E}\,\psi(u+W)-\sum_{k=1}^{m} f_k e^{-r_k u}\tilde{\mathbf{w}}(r_k)\sum_{j=1}^{m}\frac{a_j b_j}{b_j-r_k} \\ &\quad -\sum_{j=1}^{m} a_j\tilde{\mathbf{w}}(b_j)e^{-b_j u}\left[1-\sum_{k=1}^{m}\frac{b_j f_k}{b_j-r_k}\right]\end{aligned}$$

Hence

$$\begin{aligned}a_0 \mathsf{E}\,\psi(u+W)+\sum_{k=1}^{m} f_k e^{-r_k u}\tilde{\mathbf{w}}(r_k)\sum_{j=1}^{m}\frac{a_j b_j}{b_j-r_k} &= \sum_{k=1}^{m} f_k e^{-r_k u}\tilde{\mathbf{w}}(r_k)\left(a_0+\sum_{j=1}^{m}\frac{a_j b_j}{b_j-r_k}\right) \\ &= \sum_{k=1}^{m} f_k e^{-r_k u}\tilde{\mathbf{w}}(r_k)x(-r_k) \\ &= \sum_{k=1}^{m} f_k e^{-r_k u},\end{aligned}$$

and

$$\sum_{k=1}^{m}\frac{b_j f_k}{b_j-r_k} = 1, \qquad j=1,\ldots,m.$$

These conditions determine the constants $\{f_k; k=1,\ldots,m\}$, once the roots $\{r_k; k=1,\ldots,m\}$ have been found.

In the particular case where the distribution of X is a mass a_0 at 0 and otherwise an $\mathbf{exp}(b)$ distribution, we find

$$\psi(u) = \frac{b-r_1}{b}e^{-r_1 u}.$$

7. Extensions and conclusions

This paper has shown one way of stating the ruin problem when either waiting times or claims have rational Laplace transforms. When waiting times are in $\mathcal{R}_+^f$, then the Laplace transform of the probability of ruin has an explicit expression, which depends on the zeros of $d(s) = \tilde{\mathbf{w}}(-s)\tilde{\mathbf{x}}(s) - 1$ in the right half-plane. When claims are in $\mathcal{R}_+^f$, then more explicit results are obtained, in that the probability of ruin itself has a rational Laplace transform, with poles which are the zeros of $d(s) = \tilde{\mathbf{w}}(-s)\tilde{\mathbf{x}}(s) - 1$ in the left half-plane. The expressions given for the probability of ruin in Section 6 show that the zeros of $d(s)$ are unavoidable, whatever way the probability of ruin is obtained, even if they are complex. An approach based on differential equations, as used in Dickson (1998) and Dickson & Hipp (1998, 2000) will involve finding the zeros of $d(s)$ at some point, for instance when solving a differential equation with constant coefficients.

Concerning the papers Dickson (1998) and Dickson & Hipp (1998, 2000) the following points may be made (refer also to Kalashnikov (1998)):

• Those three papers assume that waiting times are rational, with two poles (combination of two exponentials, or an $\mathbf{Erlang}(2, \lambda)$). The Laplace transform of the probability of ruin is given by Theorems 5.1-5.2, by specifying that either $n = 2, d_1 = d_2 = 1$, or $n = 1, d_1 = 2$. The roots of $d(s)$ which appear in Theorem 5.2 are found in the three papers, when solving ordinary differential equations which ultimately result from Eq.(2.1).

• The examples given in the three papers all involve claims with rational distributions. In some cases the claims distribution is a combination of two exponentials, and the probability of ruin is also a combination of two exponentials; in one case the claims distribution is a combination of two $\mathbf{Erlang}(2, \lambda)$ distributions, which has thus four poles, yielding a probability of ruin which is a combination of four exponentials. All those facts are independent of the particular distribution for the waiting times, as shown in Section 6. However, the approach described in this paper avoids ladder variables and differential equations altogether.

On the subject of ladder variables (see Feller (1971, Chapter 12), Bowers *et al.* (1997, Chapter 13)), observe that once the Laplace transform of the probability of ruin and $\varphi(0)$ have been obtained (for instance, using Theorems 5.1-5.2), the Laplace transform $\tilde{\mathbf{q}}(s)$ of the ladder variable Q follows immediately from the identity

$$s\tilde{\varphi}(s) \;=\; \frac{\varphi(0)}{1 - \psi(0)\tilde{\mathbf{q}}(s)}.$$

From this point of view, once $\varphi(0)$ and $\tilde{\varphi}(s)$ are known, ladder height representations for the probability of ruin are no more than one possible way of inverting $\tilde{\varphi}(s)$.

The following claims will be substantiated in a subsequent paper:

Theorems 3.2-3.5, 6.1 and 6.2 were stated with the condition $\mathsf{h}_W < 0$*, but they also hold if* $\mathsf{h}_W = 0$. This is shown by taking appropriate limits with respect to the distribution of W. Likewise, *Theorems 5.1-5.2 hold if* $\mathsf{h}_X = 0$.

It is also possible to consider a more general class of distributions, $\mathcal{R}_+$, which consists in distributions on $\mathbb{R}_+$ with a Laplace transform which is a series of rational functions. Under proper assumptions, the density of such distributions can be expressed as a series of Erlang densities. Moreover, *Theorems 5.1, 5.2, 6.1 and 6.2 can be extended to cases where W or X are in $\mathcal{R}_+$.* In particular, it is interesting that if the claims have a density which is a series of exponentials times polynomials, then the probability of ruin is of the same type.

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