ALGEBRAIC PROPERTIES OF BETA AND GAMMA DISTRIBUTIONS, AND APPLICATIONS

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Abstract

Three new properties are derived. The first one relates to the distribution of $UG + G'$, where the three variables are independent, G, G' have gamma distributions and U is arbitrary; the second one concerns the case where U has a beta distribution, and the third one the case where $1/U$ has a beta distribution. These properties generalize the additivity of independent gamma distributions with the same scale parameter. Applications to Markov chains and to the stochastic equation $X = (\text{in law})\, U(X + C)$ are given. The last section shows how non-negative relationships may be transformed into new ones on the whole line. In particular, the limit distribution of a time series with random coefficients and Gaussian error terms is found.

Beta Distribution; Gamma Distribution; Discounted Sums;

Markov Chains; Stochastic Difference Equations;

Time Series with Random Coefficients

1. Introduction

This paper concerns three properties of beta and gamma distributions and some of their consequences. The properties are derived in Section 2. They are generalizations of the usual additivity of independent gamma distributions with the same scale parameter. Section 3 gives applications to Markov chains (not all homogeneous), and defines a *loop*, a chain which starts with a distribution μ_0 and returns to μ_0 after N steps. The properties of beta and gamma distributions yield explicit examples of loops. Finally, Section 4 shows how to transform non-negative relationships into new ones on the whole line. Applications to time series with random coefficients are given.

Many of the applications given have to do with the distribution of the sums

$$X_n \;=\; \sum_{k=0}^{n-1} C_k U_k \cdots U_{n-1}, \quad Y_n \;=\; \sum_{k=1}^{n} C_k U_1 \cdots U_k, \tag{1}$$

where $\{(U_n, C_n); n \geq 0\}$ is an i.i.d. sequence $\mathbb{R}^2$-valued random variables. For each fixed n, X_n and Y_n have the same distribution. The process $\{X_n; n \geq 0\}$ is Markovian and satisfies

$$X_{n+1} \;=\; U_n(X_n + C_n), \tag{2}$$

with X_n independent of (U_n, C_n); if $\{Y_n\}$ converges almost surely, then $\{X_n\}$ converges in distribution to the same limit X, which satisfies

$$X \;\stackrel{\mathcal{L}}{=}\; U(X + C), \quad (U, C) \text{ and } X \text{ independent.} \tag{3}$$

There is no general technique for solving (3) for X, given a joint distribution for (U, C), but a number explicit examples are known, see Dufresne (1996) and references therein (Goldie and Grübel (1990) study the tails of X). The third property of Section 2 yields a new explicit example of (3), where U and C are independent, U is distributed as the ratio of two beta variables, C has a gamma distribution, and X is distributed as the ratio of gamma and beta variables. The N-loops of Section 3 each give rise to a new examples of (3) (which is itself a 1-loop), but where U and C are dependent.

2. New properties of beta and gamma distributions

Notation. All variables denoted G_a (for some $a > 0$, with or without superscript) have a $\Gamma(a, 1)$ distribution; all variables denoted $B_{a,b}$ (for some $a, b > 0$, with or without superscript) have a $\beta_{a,b}$ distribution. In all expressions the variables G_{a_1}, $G'_{a_2}, \ldots, B_{a_1,b_1}, B'_{a_2,b_2}, \ldots$ are assumed independent. The notation "$V \sim \mathcal{L}$" stands for "the variable V has probability distribution $\mathcal{L}$". If $V_i \sim \mathcal{L}_i, i = 1, 2$, are independent, then the distribution of $V_1 + V_2$ is denoted $\mathcal{L}_1 * \mathcal{L}_2$, and the distribution of $V_1 V_2$ is denoted $\mathcal{L}_1 \odot \mathcal{L}_2$. □

An important (though elementary) property used in the sequel is

$$\frac{G_a}{G_a + G'_b} \quad \text{is independent of} \quad (G_a + G'_b), \tag{4}$$

which characterizes gamma distributions (Lukacs, 1955). An immediate consequence is

$$\beta_{a,b} \odot \Gamma(a + b, 1) \;=\; \Gamma(a, 1). \tag{5}$$

Theorem 1. *For any $a, b > 0$,*

$$UG_a + G'_b \;\stackrel{\mathcal{L}}{=}\; G''_{a+b}[1 + (U - 1)B_{a,b}], \tag{6}$$

where all the variables are assumed independent.

Proof. By the independence assumptions and property (4), we may multiply either sides of the identity

$$G''_{a+b} \stackrel{\mathcal{L}}{=} G_a + G'_b$$

by $1 + (U - 1)B_{a,b}$ to obtain

$$\begin{aligned} G''_{a+b}[1 + (U - 1)B_{a,b}] \;&=\; G''_{a+b}(1 - B_{a,b} + UB_{a,b}) \\ &\stackrel{\mathcal{L}}{=}\; G''_{a+b}\left(\frac{G'_b}{G_a + G'_b} + \frac{UG_a}{G_a + G'_b}\right) \\ &\stackrel{\mathcal{L}}{=}\; G'_b + UG_a. \end{aligned}$$

□

Example 1. If U equals 0 w.p. p and the constant u w.p. $q = 1 - p$, then (if the three variables are independent) $UG_a + G'_b$ is $\Gamma(b, 1)$ w.p. p and has the same distribution as $G''_{a+b}[1 + (u-1)B_{a,b}]$ w.p. q. In particular, the affine transformation of exponential variables $UG_1 + G'_1$ has w.p. q the same distribution as G_2 times an independent variable with a uniform distribution over the interval $(1, u)$. □

Theorem 1 yields a short proof for a generalization of Theorem 1 in Dufresne (1996).

Theorem 2. *(A) For any $a, b, c, d > 0$,*

$$B_{a,b}G_c + G'_d \overset{\mathcal{L}}{=} G''_{c+d}(1 - B_{b,a}B'_{c,d}). \tag{7}$$

(B) In particular, for any $a, b, c > 0$,

$$B_{a,b+c}G_b + G'_c \overset{\mathcal{L}}{=} G_{b+c}B_{a+c,b} \overset{\mathcal{L}}{=} G_{a+c}B_{b+c,a}. \tag{8}$$

Proof. Eq. (7) results from (6). Let $b = c + d$ in (7). Then

$$B_{b,a}B'_{c,d} = B_{c+d,a}B'_{c,d} \overset{\mathcal{L}}{=} B_{c,a+d} \tag{9}$$

which yields the first identity in (8), after substituting (b, c) for (c, d). The second identity in (9) may be justified using Mellin transforms, or else as follows:

$$\begin{aligned}\frac{G_{c+d}}{G_{c+d} + G'_a} \cdot \frac{G''_c}{G''_c + G'''_d} &= g_1(G_{c+d}, G'_a) \cdot g_2\left(\frac{G''_c}{G''_c + G'''_d}\right) \\ &\overset{\mathcal{L}}{=} g_1(G''_c + G'''_d, G_a) \cdot g_2\left(\frac{G''_c}{G''_c + G'''_d}\right) \\ &= \frac{G''_c}{G''_c + G'''_d + G_a} \sim \beta_{c,a+d}.\end{aligned}$$

The last identity in (8) is obtained from Mellin transforms:

$$\begin{aligned}\mathsf{E}(G_{b+c}B_{a+c,b})^t &= \frac{\Gamma(b+c+t)}{\Gamma(b+c)} \cdot \frac{\Gamma(a+c+t)}{\Gamma(a+c)} \cdot \frac{\Gamma(a+b+c)}{\Gamma(a+b+c+t)} \\ &= \mathsf{E}(G_{a+c}B_{b+c,a})^t.\end{aligned}$$

□

Theorem 3. *(A) For any $a, b, c, d > 0$,*

$$\frac{G_a}{B_{c,d}} + G'_b \overset{\mathcal{L}}{=} G_{a+b}\left(\frac{B_{a,b}}{B'_{c,d}} + 1 - B_{a,b}\right) \overset{\mathcal{L}}{=} G_{a+b}\left(1 + B_{a,b}\frac{G'_d}{G''_c}\right). \tag{10}$$

(B) In particular, for any $a, b, c > 0$,

$$\frac{G_a}{B_{b,a+c}} + G'_c \overset{\mathcal{L}}{=} \frac{G_{a+c}}{B_{b,a}}. \tag{11}$$

Proof. Eqs. (10) directly follow from (6), upon noting that

$$B_{a,b}\left(\frac{1}{B'_{c,d}} - 1\right) \overset{\mathcal{L}}{=} B_{a,b}\left(\frac{G''_c + G'_d}{G''_c} - 1\right).$$

When $d = a + b$, $B_{a,b}G'_d/G''_c \overset{\mathcal{L}}{=} G'''_a/G^{(4)}_c$, which yields (11) after reversing the roles of b and c. □

Theorem 4. *The unique solution X of (3) with $U \overset{\mathcal{L}}{=} B_{a,c}/B'_{a+b,c}$ independent of $C \sim \Gamma(c, 1)$ has the same distribution as $G_a/B_{b,a+c}$. This means that, if $\{(U_n, C_n); n \geq 1\}$ are independent copies of (U, C),*

$$\sum_{k\geq 1} C_k U_1 \cdots U_k \overset{\mathcal{L}}{=} \frac{G_a}{B_{b,a+c}}.$$

Proof. Multiplying (11) by $B'_{a,c}/B''_{a+b,c}$ results in

$$\begin{aligned} \frac{G_{a+c}}{B_{b,a}} \cdot B'_{a,c}/B''_{a+b,c} &\overset{\mathcal{L}}{=} G_{a+c} \cdot \frac{G'_b + G''_a}{G'_b} \cdot \frac{G'''_a}{G'''_a + G^{(4)}_c} \cdot \frac{G^{(5)}_{a+b} + G^{(6)}_c}{G^{(5)}_{a+b}} \\ &\overset{\mathcal{L}}{=} G_a \cdot \frac{G'_a + G''_b + G'''_c}{G''_b}. \end{aligned}$$

(This may also be checked using Mellin transforms.) Existence and uniqueness of the solution follow from

$$\begin{aligned} \mathsf{E}\log U &= \mathsf{E}\log B_{a,c} - \mathsf{E}\log B'_{a+b,c} \\ &= \mathsf{E}\left[\log\left(1 + \frac{G_c}{G'_a + G''_b}\right) - \log\left(1 + \frac{G_c}{G'_a}\right)\right] < 0. \end{aligned}$$

□

3. Applications to positive Markov chains and loops

In this paper, the Mellin transform of a probability distribution μ, when it exists, is defined as the mapping

$$t \mapsto \int_{\mathbb{R}} x^t \, \mu(dx).$$

This is the form seen most often in probability, although in analysis "x^t" is usually replaced with "x^{t-1}"; compare Davies (1985) and Chamayou and Letac (1991). Our results only require $t \in \mathbb{R}$, though the general theory of Mellin transforms is formulated for complex arguments.

Notation. The distribution with Mellin transform

$$\frac{\Gamma(a_1+t)\cdots\Gamma(a_p+t)}{\Gamma(a_1)\cdots\Gamma(a_p)} \times \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(b_1+t)\cdots\Gamma(b_q+t)} \times \frac{\Gamma(c_1-t)\cdots\Gamma(c_r-t)}{\Gamma(c_1)\cdots\Gamma(c_r)} \times \frac{\Gamma(d_1)\cdots\Gamma(d_r)}{\Gamma(d_1-t)\cdots\Gamma(d_r-t)}$$

(if it exists) will be denoted

$$\left(\begin{matrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_q \end{matrix} \,\middle|\, \begin{matrix} c_1 & \cdots & c_r \\ d_1 & \cdots & d_s \end{matrix}\right). \tag{12}$$

When there are no c's or d's this is abbreviated as follows:

$$\left(\begin{matrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_q \end{matrix}\right) := \left(\begin{matrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_q \end{matrix} \,\middle|\, \begin{matrix} - \\ - \end{matrix}\right). \qquad \square$$

The law (12) exists, for instance, if $p \geq q$, $a_i < b_i$, $1 \leq i \leq q$, $r \geq s$, $c_i < d_i$, $1 \leq i \leq s$, in which case it is the same as the ratio of two independent variables with distributions

$$\beta_{a_1,b_1-a_1} \odot \cdots \odot \beta_{a_q,b_q-a_q} \odot \Gamma(a_{q+1},1) \odot \cdots \odot \Gamma(a_p,1)$$

and

$$\beta_{c_1,d_1-c_1} \odot \cdots \odot \beta_{c_s,d_s-c_s} \odot \Gamma(c_{s+1},1) \odot \cdots \odot \Gamma(c_r,1).$$

Remark. The above notation simplifies the application of the properties of Section 2, as the examples below will show. There are two obvious rules in using symbols (12). The first is that parameters which appear "above" and "below" may be cancelled, for instance if $c_1 = d_2$ then the two parameters may be omitted (this is what happens in Eq. (5)). The second is for the multiplication of independent variables:

$$\left(\begin{matrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_q \end{matrix}\middle| \begin{matrix} c_1 & \cdots & c_r \\ d_1 & \cdots & d_s \end{matrix}\right) \odot \left(\begin{matrix} a'_1 & \cdots & a'_{p'} \\ b'_1 & \cdots & b'_{q'} \end{matrix}\middle| \begin{matrix} c'_1 & \cdots & c'_{r'} \\ d'_1 & \cdots & d'_{s'} \end{matrix}\right)$$

$$= \left(\begin{matrix} a_1 & \cdots & a_p & a'_1 & \cdots & a'_{p'} \\ b_1 & \cdots & b_q & b'_1 & \cdots & b'_{q'} \end{matrix}\middle| \begin{matrix} c_1 & \cdots & c_r & c'_1 & \cdots & c'_{r'} \\ d_1 & \cdots & d_s & d'_1 & \cdots & d'_{s'} \end{matrix}\right). \quad \square$$

Suppose $\{(U_n, C_n)\}$ and X_0 are independent and consider the Markov chain (2) those variables induce.

Definition. $\{(U_n, C_n), n = 0, 1, \dots N-1; \mu_0\}$ *is said to be an* **N-loop** *if*

$$\mathcal{L}(X_0) = \mu_0 \quad \Longrightarrow \quad \mathcal{L}(X_N) = \mathcal{L}(X_0).$$

Clearly Eq. (3) is the 1-loop $\{(U, C); \mathcal{L}(X)\}$. It is also obvious that, for any $k = 2, 3, \dots$, an N-loop becomes a kN-loop by repeating the $\{(U_n, C_n); n = 0, 1, \dots N-1\}$ $k-1$ times.

Theorem 5. *Suppose* $\{(U_n, C_n), n = 0, 1, \dots N-1; \mu_0\}$ *is an* N*-loop. Let* Y_0, $\{(V_n, D_n), n = 0, 1, \dots\}$ *be independent with* (V_n, D_n) *having the same joint distribution as*

$$V = U_0 \cdots U_{N-1}, \quad D = U_0 \cdots U_{N-1} C_0 + U_1 \cdots U_{N-1} C_1 + \cdots + U_{N-1} C_{N-1}.$$

Then the Markov chain

$$Y_{n+1} = V_n Y_n + D_n$$

has stationary distribution μ_0*, that is to say* $\mathcal{L}(Y) = \mu_0$ *implies*

$$Y \stackrel{\mathcal{L}}{=} VY + D, \quad Y \text{ and } (V, D) \text{ independent.}$$

Remark. If $\mathsf{P}(U_0 \cdots U_{N-1} = 0) = 0$, it is of course possible to define $\tilde{D} = D/U_0 \cdots U_{N-1}$ and to rewrite the last equation as

$$Y \stackrel{\mathcal{L}}{=} V(Y + \tilde{D}), \quad Y \text{ and } (V, \tilde{D}) \text{ independent.} \qquad \square$$

Example 2. Let $a_0, \ldots, a_N, b_1, \ldots, b_N > 0$ be such that $a_n < a_{n+1} + b_{n+1}$ for $n = 0, \ldots, N-1$. Let $X_0 \sim \Gamma(a_0, 1)$, $C_n \sim \Gamma(a_{n+1} + b_{n+1} - a_n, 1)$, $U_n \sim \beta_{a_{n+1}, b_{n+1}}$ be independent. Then recursion (2) implies

$$\begin{aligned} X_0 + C_0 &\sim \begin{pmatrix} a_1 + b_1 \\ - \end{pmatrix}, \quad X_1 \sim \begin{pmatrix} a_1 & a_1 + b_1 \\ & a_1 + b_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ - \end{pmatrix} \\ X_1 + C_1 &\sim \begin{pmatrix} a_2 + b_2 \\ - \end{pmatrix}, \quad X_2 \sim \begin{pmatrix} a_2 & a_2 + b_2 \\ & a_2 + b_2 \end{pmatrix} = \begin{pmatrix} a_2 \\ - \end{pmatrix} \end{aligned}$$

and so $X_n \sim \Gamma(a_n, 1)$ for all $n = 0, \ldots, N$. If $a_N = a_0$, then $\{(U_n, C_n), n = 0, 1, \ldots N-1; \Gamma(a_0, 1)\}$ is an N-loop. In this case the variable V of Theorem 5 has a $\beta_{a_1,b_1} \odot \cdots \odot \beta_{a_N,b_N}$ distribution, which simplifies to $\beta_{a_1, a_N + b_N}$ if we furthermore have $a_{n+1} = a_n + b_n, n = 1, \ldots, N-1$. $\square$

Example 3. Let $\{a_n; n \geq 0\}$ and $\{b_n; n \geq 0\}$ be strictly positive numbers, with $a_n < b_n \leq b_{n+1}$, $a_{n+1} \leq b_n - a_n$ for all $n \geq 0$. Assume the independent variables X_0, $\{(U_n, C_n); n = 0, 1, \ldots\}$ have distributions

$$X_0 \sim \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}, \quad -U_n \sim \begin{pmatrix} a_{n+1} & b_n \\ b_n - a_n & b_{n+1} \end{pmatrix}, \quad C_n \equiv -1.$$

Then $X_n \sim \begin{pmatrix} a_n \\ b_n \end{pmatrix}$ for $n \geq 0$. A loop is possible only if $b_n \equiv b$, which means that $-U_n$ is a single beta variable (not the product of two), and if $a_N = a_0$ for some $N \geq 1$. When $N = 1$ we get an explicit solution of (3) due to Letac (1986):

$$X \sim \begin{pmatrix} a \\ b \end{pmatrix}, \quad -U \sim \begin{pmatrix} a \\ b - a \end{pmatrix}, \quad C \equiv -1.$$

This example is not far removed from the problem considered by Chamayou (1996), which is to find the stationary distribution of the Markov chain $X_{n+1} = U_n(1 - X_n)$, where X_n is independent of $U_n \sim \beta_{a,1} \odot \beta_{a,1}$. No explicit result has been found in that case. $\square$

Example 4. Let $\{a_n; n \geq 0\}$ and $\{b_n; n \geq 0\}$ be strictly positive numbers, with $b_{n+1} \leq b_n$ for all $n \geq 0$. Assume the independent variables X_0, $\{(U_n, C_n); n = 0, 1, \ldots\}$ have distributions

$$X_0 \sim \left(\begin{matrix} a_0 \\ - \end{matrix} \,\middle|\, \begin{matrix} b_0 \\ - \end{matrix} \right), \quad U_n \sim \left(\begin{matrix} a_{n+1} \\ - \end{matrix} \,\middle|\, \begin{matrix} a_n + b_n & b_{n+1} \\ b_n & \end{matrix} \right), \quad C_n \equiv 1.$$

From the trivial identity

$$\left(\begin{matrix} a \\ - \end{matrix} \,\middle|\, \begin{matrix} b \\ - \end{matrix} \right) * \delta_1 \; = \; \left(\begin{matrix} - \\ - \end{matrix} \,\middle|\, \begin{matrix} b \\ a + b \end{matrix} \right)$$

(where δ_1 is the Dirac measure at the point 1) we get $X_n \sim \left(\begin{matrix} a_n \\ - \end{matrix} \,\middle|\, \begin{matrix} b_n \\ - \end{matrix} \right)$ for $n \geq 0$. A loop is possible only if $b_n \equiv b$, in which case both X_n and U_n have a "beta distribution of the second kind" (i.e. the same law as the ratio of two independent gamma variables):

$$X_0 \sim \left(\begin{matrix} a_0 \\ - \end{matrix} \,\middle|\, \begin{matrix} b \\ - \end{matrix} \right), \quad U_n \sim \left(\begin{matrix} a_{n+1} \\ - \end{matrix} \,\middle|\, \begin{matrix} a_n + b \\ - \end{matrix} \right), \quad C_n \equiv 1, \tag{13}$$

and $X_N \stackrel{\mathcal{L}}{=} X_0$ if $a_N = a_0$. This loop is due to Chamayou and Letac (1991) and Goldie (1991). $\square$

Eqs. (8) and (11) (Theorems 2 and 3) may be expressed as

Theorem 2 (B)

$$\left(\begin{matrix} a & b \\ \multicolumn{2}{c}{a + b + c} \end{matrix} \right) * \left(\begin{matrix} c \\ - \end{matrix} \right) \; = \; \left(\begin{matrix} a + c & b + c \\ \multicolumn{2}{c}{a + b + c} \end{matrix} \right) \tag{14}$$

Theorem 3 (B)

$$\left(\begin{matrix} a \\ - \end{matrix} \,\middle|\, \begin{matrix} b \\ a + b + c \end{matrix} \right) * \left(\begin{matrix} c \\ - \end{matrix} \right) \; = \; \left(\begin{matrix} a + c \\ - \end{matrix} \,\middle|\, \begin{matrix} b \\ a + b \end{matrix} \right). \tag{15}$$

These identities lead to other examples of loops and non-homogeneous Markov chains.

Example 5. Let $\{a_n; n \geq 0\}$, $\{b_n; n \geq 0\}$ and $\{c_n; n \geq 0\}$ be strictly positive numbers. Assume the independent variables X_0, $\{(U_n, C_n); n = 0, 1, \ldots\}$ have distributions

$$X_0 \sim \begin{pmatrix} a_0 & b_0 \\ \multicolumn{2}{c}{a_0 + b_0 + c_0} \end{pmatrix}, \quad C_n \sim \begin{pmatrix} c_n \\ - \end{pmatrix}$$

$$U_n \sim \begin{pmatrix} a_{n+1} & b_{n+1} & a_n + b_n + c_n \\ a_n + c_n & b_n + c_n & a_{n+1} + b_{n+1} + c_{n+1} \end{pmatrix}.$$

Let $X_{n+1} = U_n(X_n + C_n)$. It can then be verified by induction that

$$X_n \sim \begin{pmatrix} a_n & b_n \\ \multicolumn{2}{c}{a_n + b_n + c_n} \end{pmatrix}.$$

If, for some N, $a_N = a_0$, $b_N = b_0$ and $c_N = c_0$, or if $a_N = b_0$, $b_N = a_0$ and $c_N = c_0$, then $\mathcal{L}(X_N) = \mathcal{L}(X_0)$.

There are various interesting special cases of this example, two of which are described below. □

Example 6. In example 5, if $a_n + b_n + c_n$ is constant for all n then U_n is the product of two independent beta variables, instead of three. If, furthermore, $b_{n+1} = a_n + c_n$ then U_n is just one beta variable. These conditions hold if $a_0, b_0, c_0, \ldots, c_N > 0$ are such that $a_0 > c_1 + \ldots + c_N$. We then define $a_n = a_0 - (c_1 + \ldots + c_n)$, $b_n = b_0 + c_0 + \ldots + c_{n-1}$ and $s = a_n + b_n + c_n$ (which does not depend on n). If the independent variables X_0, $\{(U_n, C_n); n = 0, \ldots, N-1\}$ have distributions

$$X_0 \sim \begin{pmatrix} a_0 & & b_0 \\ & s & \end{pmatrix}, \quad C_n \sim \begin{pmatrix} c_n \\ - \end{pmatrix}, \quad U_n \sim \begin{pmatrix} a_{n+1} \\ a_n + c_n \end{pmatrix}.$$

then

$$X_n \sim \begin{pmatrix} a_n & & b_n \\ & s & \end{pmatrix}, \quad n \leq N. \tag{16}$$

If, moreover, $a_N = b_0$ and $b_N = a_0$ then $\mathcal{L}(X_N) = \mathcal{L}(X_0)$. □

Example 7. In Example 5 it is also possible to have the U's distributed as a single beta variable by letting $a_n + b_n + c_n$ be constant and arranging for $a_{n+1} = b_n + c_n$, instead of $b_{n+1} = a_n + c_n$. Here is an example: let $b_0, b_1, c_0, c_1 > 0$, $s = b_0 + b_1 + c_0 + c_1$, and suppose X_0, U_0, C_0, U_1, C_1 are independent variables with distributions

$$X_0 \sim \begin{pmatrix} b_1 + c_1 & & b_0 \\ & s & \end{pmatrix}, \quad C_n \sim \begin{pmatrix} c_n \\ - \end{pmatrix}, \quad n = 0, 1$$

$$U_0 \sim \begin{pmatrix} b_1 \\ b_1 + c_0 + c_1 \end{pmatrix}, \quad U_1 \sim \begin{pmatrix} b_0 \\ b_0 + c_0 + c_1 \end{pmatrix}.$$

(Here the parameters a_0, a_1 of Example 5 are replaced with $b_1 + c_1, b_0 + c_0$, respectively.) Then

$$X_0 + C_0 \sim \begin{pmatrix} b_1 + c_0 + c_1 & & b_0 + c_0 \\ & s & \end{pmatrix}$$

$$X_1 \sim \begin{pmatrix} b_1 & & b_0 + c_0 \\ & s & \end{pmatrix}, \quad X_1 + C_1 \sim \begin{pmatrix} b_1 + c_1 & & b_0 + c_0 + c_1 \\ & s & \end{pmatrix}$$

and $\mathcal{L}(X_2) = \mathcal{L}(X_0)$. It is possible to create other loops (with $N \geq 2$) where the U's are single beta variables by alternating the simplifications $a_{n+1} = b_n + c_n$ and $b_{n+1} = a_n + c_n$ in Example 5. $\square$

Example 8. Let $\{a_n; n \geq 0\}$, $\{b_n; n \geq 0\}$ and $\{c_n; n \geq 0\}$ be strictly positive numbers, and let the following variables be independent:

$$X_0 \sim \left(\begin{array}{c|c} a_0 & b_0 \\ - & a_0 + b_0 + c_0 \end{array}\right), \quad C_n \sim \begin{pmatrix} c_n \\ - \end{pmatrix}$$

$$U_n \sim \left(\begin{array}{c|cc} a_{n+1} & a_n + b_n & b_{n+1} \\ a_n + c_n & a_{n+1} + b_{n+1} + c_{n+1} & b_n \end{array}\right).$$

Then

$$X_0 + C_0 \sim \left(\begin{array}{c|c} a_0 + c_0 & b_0 \\ - & a_0 + b_0 \end{array}\right), \quad X_1 \sim \left(\begin{array}{c|c} a_1 & b_1 \\ - & a_1 + b_1 + c_1 \end{array}\right),$$

and so on, and in general

$$X_n \sim \left(\begin{array}{c|c} a_n & b_n \\ - & a_n + b_n + c_n \end{array}\right).$$

If $a_N = a_0$, $b_N = b_0$ and $c_N = c_0$ for some $N \geq 1$ then $\mathcal{L}(X_N) = \mathcal{L}(X_0)$. Observe that the case $N = 1$ corresponds to Theorem 4, where the distribution of U_n simplifies to the ratio of a beta to just one other beta (instead of two). For some sequences $\{a_n\}$ and $\{c_n\}$ this simplification is also possible, if we can construct $\{b_n\}$ so that

$$a_n + b_n \;=\; a_{n+1} + b_{n+1} + c_{n+1}, \quad b_n \;>\; 0 \quad \text{for all } n.$$

In that case we must have

$$b_n \;=\; b_0 + a_0 - a_n - (c_1 + \cdots + c_n),$$

a relationship which implies that no loop is possible. □

Example 9. Examples 2 to 6 and 8 also yield limit distributions for some non-homogeneous Markov chains. For instance, consider Example 6, where a_0 and $\{c_n; n \geq 0\}$ are strictly positive and such that $a_0 < \sum_{n\geq 1} c_n$; the distribution of X_0 and the sequences $\{a_n\}$ and $\{b_n\}$ are defined as before. The non-homogeneous Markov chain $\{X_n; n \geq 0\}$ then has marginals (16) and limit distribution

$$\begin{pmatrix} a_0 - \sum\limits_{n\geq 1} c_n \qquad b_0 + \sum\limits_{n\geq 0} c_n \\ a_0 + b_0 + c_0 \end{pmatrix}. \qquad \square$$

4. Properties and Markov chains on the whole line

Notation. All variables denoted $\overline{G}_a$ (for some $a > 0$, with or without superscript) have the same distribution as $G_a - G'_a$; this distribution will be called *double-gamma* with parameter a, and will be denoted $\mathrm{D}\Gamma(a, 1)$. All variables denoted Z (with or without superscript) have a standard normal distribution. In all expressions the variables $\overline{G}^{(1)}_{a_1}$, $\overline{G}^{(2)}_{a_2}$, …(respectively $Z^{(1)}$, $Z^{(2)}$, …) are assumed independent. □

Remark. The double-gamma distribution just introduced inherits the additivity of gamma distributions:

$$\overline{G}_a + \overline{G}'_b \;\stackrel{\mathcal{L}}{=}\; G_a + G'_b - (G''_a + G'''_b) \;\stackrel{\mathcal{L}}{=}\; \overline{G}_{a+b},$$

or $\mathrm{D}\Gamma(a,1) * \mathrm{D}\Gamma(b,1) = \mathrm{D}\Gamma(a+b,1)$. The density g_a of $\overline{G}_a$ may be expressed in terms of special functions:

$$\begin{aligned} g_a(u) &= e^{-|u|}\int_0^\infty (y+|u|)^{a-1}y^{a-1}e^{-2y}\,dy/\Gamma(a)^2 \\ &= 2^{-a}|u|^{a-1}e^{-|u|}\int_0^\infty \left(1+\frac{t}{2|u|}\right)^{a-1} t^{a-1}e^{-t}\,dt/\Gamma(a)^2 \\ &= \frac{2^{-a+\frac{1}{2}}|u|^{a-\frac{1}{2}}}{\sqrt{\pi}\Gamma(a)}K_{a-\frac{1}{2}}(|u|), \quad u\in\mathbb{R}, \end{aligned}$$

where

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \frac{e^{-z}}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\infty e^{-s}s^{\nu-\frac{1}{2}}\left(1+\frac{s}{2z}\right)^{\nu-\frac{1}{2}} ds$$

($\Re(\nu) > -1/2$, $|\arg z| < \pi$) is *MacDonald's function* (Lebedev, 1972, p.140). When a is an integer this simplifies to

$$g_n(u) = \frac{e^{-|u|}}{\Gamma(n)}\sum_{j=0}^{n-1}\frac{\Gamma(n+j)}{2^{n+j}\Gamma(j+1)\Gamma(n-j)}|u|^{n-1-j}, \quad n=1,2,\ldots \tag{17}$$

This equation shows that the absolute value of a $\mathrm{D}\Gamma(n,1)$ variable is a mixture of $\Gamma(1,1),\ldots,\Gamma(n,1)$ distributions, with a density equal to $2g_n(u)\mathbf{1}_{(0,\infty)}(u)$. □

Theorem 6. *Suppose* $\{G^{(j)}_{a_j}; j=0,1,\ldots,n\}$ *are independent of the non-negative variables* $\{V^{(j)}; j=0,1,\ldots,n\}$, *and also that*

$$G^{(0)}_{a_0}V^{(0)} \stackrel{\mathcal{L}}{=} \sum_{j=1}^n G^{(j)}_{a_j}V^{(j)}. \tag{18}$$

Then

$$\overline{G}^{(0)}_{a_0}\sqrt{V^{(0)}} \stackrel{\mathcal{L}}{=} \sum_{j=1}^n \overline{G}^{(j)}_{a_j}\sqrt{V^{(j)}}, \tag{19}$$

where $\{\overline{G}^{(j)}_{a_j}; j=0,1,\ldots,n\}$ *and* $\{V^{(j)}; j=0,1,\ldots,n\}$ *are independent, the joint law of the* V*'s being unchanged.*

The result also holds if the sums on the right-hand sides of (18) and (19) are infinite.

Proof. For $n < \infty$ we have (the case of infinite sums is similar)

$$\begin{aligned}
\mathsf{E}\exp\left(it\sum_{j=1}^{n}\overline{G}_{a_j}^{(j)}\sqrt{V^{(j)}}\right) &= \mathsf{E}\mathsf{E}\left[\exp\left(it\sum_{j=1}^{n}\overline{G}_{a_j}^{(j)}\sqrt{V^{(j)}}\right)\middle| V^{(1)},\ldots,V^{(n)}\right] \\
&= \mathsf{E}\prod_{j=1}^{n}\left(1+t^2V^{(j)}\right)^{-a_j} \\
&= \mathsf{E}\exp\left(-t^2\sum_{j=1}^{n}G_{a_j}^{(j)}V^{(j)}\right) \\
&= \mathsf{E}\exp\left(-t^2G_{a_0}^{(0)}V^{(0)}\right) \\
&= \mathsf{E}\exp\left(it\overline{G}_{a_0}^{(0)}\sqrt{V^{(0)}}\right).
\end{aligned}$$

□

Theorem 6 applies to Eq. (5) and to Theorems 1, 2 and 3.

Example 10. Eq. (5) yields

$$\sqrt{\beta_{a,b}}\odot \mathrm{D}\Gamma(a+b,1) \;=\; \mathrm{D}\Gamma(a,1).$$

(where $\sqrt{\beta_{a,b}} = \mathcal{L}(\sqrt{B_{a,b}})$). In particular, since $\sqrt{\beta_{a,1}} = \beta_{2a,1}$, this implies

$$\beta_{2a,1}\odot \mathrm{D}\Gamma(a+1,1) \;=\; \mathrm{D}\Gamma(a,1).$$

If we take absolute values on both sides and let a be an integer, we get a relationship for the product of a beta and an independent mixture of gamma variables (see Eq. (17)). For instance, the product of a $\beta_{2,1}$ and the absolute value of an independent $\mathrm{D}\Gamma(2,1)$, which has density $e^{-u}(1+u)/2$, has an exponential distribution $\Gamma(1,1)$. □

Theorem 7. *Add to the assumptions of Theorem 5 that the non-negative $\{U_m\}$ are independent of the independent set $\{C_n\}$, $C_n \sim \Gamma(c_n,1)$, and that $\mathcal{L}(Y) = \mu_0 = \mathcal{L}(R)\odot\Gamma(a,1)$, $R\geq 0$. Then the Markov chain*

$$W_{n+1} \;=\; \sqrt{V_n}W_n + F_n$$

has stationary distribution $\nu_0 = \mathcal{L}(\sqrt{R}) \odot \mathrm{D}\Gamma(a,1)$, *that is to say* $\mathcal{L}(W) = \nu_0$ *implies*

$$W \stackrel{\mathcal{L}}{=} \sqrt{V}W + F, \quad W \text{ and } (V,F) \text{ independent,}$$

where $\{(V_n, F_n)\}$ *are independent copies of*

$$(V,F) \stackrel{\mathcal{L}}{=} \left(U_0 \cdots U_{N-1}, \overline{G}^{(0)}_{c_0}\sqrt{U_0\cdots U_{N-1}} + \overline{G}^{(1)}_{c_1}\sqrt{U_1\cdots U_{N-1}} + \cdots + \overline{G}^{(N-1)}_{c_{N-1}}\sqrt{U_{N-1}}\right).$$

In the last expression the two sets of variables $\{U_m\}$ *and* $\{\overline{G}^{(n)}_{c_n}\}$ *are independent.*

Theorem 7 applies to Examples 2 and 5 to 8. The following theorems are related to Example 3.7 in Vervaat (1979).

Theorem 8. *Suppose* $\{V^{(j)}; j = 0,1,\ldots,n\}$, *are non-negative and satisfy*

$$V^{(0)} \stackrel{\mathcal{L}}{=} \sum_{j=1}^{n} V^{(j)}. \tag{20}$$

Then

$$Z^{(0)}\sqrt{V^{(0)}} \stackrel{\mathcal{L}}{=} \sum_{j=1}^{n} Z^{(j)}\sqrt{V^{(j)}}, \tag{21}$$

where $\{Z^{(j)}; j = 0,1,\ldots,n\}$ *are independent of* $\{V^{(j)}; j = 0,1,\ldots,n\}$, *the joint law of the* V*'s being unchanged.*

The result also holds if the sums on the right-hand sides of (20) and (21) are infinite.

Theorem 9. *Add to the assumptions of Theorem 5 that* $U_n \geq 0$, $C_n \geq 0$ *for* $n = 0,\ldots,N-1$, *and that* $\mathcal{L}(Y) = \mu_0$ *is concentrated on* $\mathbb{R}_+$. *Then the Markov chain*

$$W_{n+1} = \sqrt{V_n}W_n + F_n$$

has stationary distribution $\nu_0 = \mathcal{L}(\sqrt{Y}) \odot \mathbf{N}(0,1)$, *that is to say* $\mathcal{L}(W) = \nu_0$ *implies*

$$W \stackrel{\mathcal{L}}{=} \sqrt{V}W + F, \quad W \text{ and } (V,F) \text{ independent,}$$

where $\{(V_n, F_n)\}$ *are independent copies of*

$$(V,F) \stackrel{\mathcal{L}}{=} \Big(U_0 \cdots U_{N-1},$$

$$Z^{(0)}\sqrt{U_0 \cdots U_{N-1}C_0} + Z^{(1)}\sqrt{U_1 \cdots U_{N-1}C_1} + \cdots + Z^{(N-1)}\sqrt{U_{N-1}C_{N-1}}\Big).$$

In the last expression the two sets of variables $\{(U_m, C_m)\}$ *and* $\{Z^{(n)}\}$ *are independent.*

Theorems 8 and 9 apply to Examples 2 and 4 to 8. In particular, we get results on some autoregressive processes with random coefficients.

Example 11. Eq. (5) implies

$$Z^{(0)}\sqrt{G_a} \stackrel{\mathcal{L}}{=} \sqrt{B_{a,b}}\left(Z^{(1)}\sqrt{G_a} + Z^{(2)}\sqrt{G'_b}\right)$$

where all the variables are independent. Thus the time series with random coefficients

$$W_{n+1} = U_n W_n + F_n, \quad n = 0, 1, \ldots,$$

with $U_m \stackrel{\mathcal{L}}{=} (B_{a,b})^{1/2}$, $F_n \stackrel{\mathcal{L}}{=} Z(G_b)^{1/2}$ independent for all (m,n), has limit distribution

$$W \stackrel{\mathcal{L}}{=} Z^{(1)}\sqrt{G_a} + Z^{(2)}\sqrt{G'_b}.$$

Observe that if $a = b = 1/2$ then this may be reexpressed as follows: if $U_m \stackrel{\mathcal{L}}{=} \left(B_{\frac{1}{2},\frac{1}{2}}\right)^{1/2}$, $F_n \stackrel{\mathcal{L}}{=} Z^{(1)}Z^{(2)}$ independent for all (m,n), then $\{W_n\}$ has limit distribution

$$W \stackrel{\mathcal{L}}{=} Z^{(1)}Z^{(2)} + Z^{(3)}Z^{(4)}.$$

Equivalently, this says that

$$\sum_{k=1}^{\infty}\left(B^{(1)}_{\frac{1}{2},\frac{1}{2}} \cdots B^{(k)}_{\frac{1}{2},\frac{1}{2}}\right)^{1/2} Z^{(2k-1)}Z^{(2k)} \stackrel{\mathcal{L}}{=} Z^{(1)}Z^{(2)}.$$ □

Example 12. The 1-loop in Example 4 (see Eq. (13)) yields the identity

$$Z^{(0)}\sqrt{G_a/G'_b} \stackrel{\mathcal{L}}{=} \sqrt{G''_a/G'''_{a+b}}\left(Z^{(1)}\sqrt{G_a/G'_b} + Z^{(2)}\right)$$

where all the variables are independent. Thus the time series with random coefficients and Gaussian errors

$$W_{n+1} \;=\; U_n W_n + Z_n, \quad n = 0, 1, \ldots,$$

with $U_m \stackrel{\mathcal{L}}{=} (G_a/G'_{a+b})^{1/2}$, $Z_n \sim \mathbf{N}(0,1)$ independent over all (m,n), has limit distribution

$$W \stackrel{\mathcal{L}}{=} Z^{(1)}\sqrt{G_a/G'_b} + Z^{(2)}.$$

As a final curiosity, observe that if $a = b = 1/2$ then

$$W \stackrel{\mathcal{L}}{=} \frac{Z^{(1)}Z^{(2)}}{Z^{(3)}} + Z^{(4)}.$$

Equivalently, this says that if $U_n \sim \left(\begin{array}{c|c} \frac{1}{2} & 1 \\ - & - \end{array}\right)$ are independent then

$$\sum_{k=1}^{\infty} (U_1 \cdots U_k)^{1/2}\, Z^{(k)} \stackrel{\mathcal{L}}{=} \frac{Z^{(1)}Z^{(2)}}{Z^{(3)}}. \qquad \square$$

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