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Weak convergence of random growth processes with applications to insurance *

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The paper is concerned with assets accumulating or discounting processes, and their weak convergence when payments are made more and more frequently during each time period. The results are applied to the calculation of the moments of actuarial functions (annuities-certain, life annuities and life insurances). The relationship with random population growth is also briefly discussed.

Keywords: Weak convergence, Actuarial functions, Population growth.

1. Introduction

In economics and finance, random rates of growth have been used to model a great variety of phenomena: stock prices, inflation, labor growth, etc. [for a survey of these models, see chapters 3 and 4 of Malliaris and Brock (1982)]. Over the last two decades, this popularity has spread to actuarial science, mainly with respect to rates of return on assets. Actuarial functions including random rates of return have been studied by J.H. Pollard (1971), Wilkie (1976, 1986, 1987), Boyle (1976), Waters (1978), Panjer and Bellhouse (1980), Bellhouse and Panjer (1981), Westcott (1981), De Jong (1984), Devolder (1986), Giacotto (1986) and Ramsay (1986). Applications to insurance include Beekman (1973), Emmanuel et al. (1975), Beekman and Fuelling (1977), Schnieper (1983) and Braun (1986). A small number of applications to pension funding have also appeared: Dufresne (1986b, 1988a,b) and O'Brien (1987).

The first part of the paper concerns the weak convergence of discrete-time processes subject to white noise growth rates, when growth takes place more and more frequently during each time interval. The second part of the paper has to do with the calculation of the moments of the limit processes. Special emphasis is placed on the duality between accumulating and discounting, and its correct interpretation when dealing with white noise rates of return.

Consider an asset (or collection of assets) having value P_s at time s. The arithmetic rate of return on this asset during period $(t-1, t)$ is

$$R_t = P_t/P_{t-1} - 1, \quad t = 1, 2, \ldots$$

The logarithmic (or geometric) rate of return over the same period is

$$\gamma_t = \log(1 + R_t) = \log(P_t/P_{t-1}), \quad t = 1, 2, \ldots$$

The sequence $\{R_t, t \geq 1\}$ will nearly always be supposed i.i.d. (which is the same as saying that $\{\gamma_t, t \geq 1\}$ is i.i.d.). The only exception is Section 8.2, where $\{\gamma_t\}$ is supposed AR(1).

The independence of successive rates of return may rightly be questionned. It can be justified on theoretical grounds, using arbitrage arguments [Samuelson (1973), Brockett (1987)]. Nevertheless, empirical studies often reject it [Panjer and Bellhouse (1980), Perry (1982), S.J. Taylor (1982)]. Leaving this controversial question aside there are two reasons for concentrating on i.i.d. growth rates. The first one is tractability. Independent growth rates are important in that they lead to explicit answers in a great variety of situations. For example, the celebrated Black–Scholes formula for the price of an option [Black and Scholes (1973), Cox et al. (1979)] is based on white noise rates of return on stocks. Secondly, continuous processes involving white noise growth rates require careful handling and interpretation, especially when it comes to setting up the basic equations. The understanding of these continuous processes is made easier (in my opinion) when they are seen as limits of discrete-time processes.

The weak convergence problem is motivated in Sections 2 and 3, and then solved in Section 4. The moments of accumulating and discounting

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processes are calculated in Section 5; this leads to applications to actuarial life functions in Section 6. Section 7 discusses the results in relation to previously published work. Finally, Section 8 translates some of the results into the language of population theory.

2. The problem

2.1. In the theory of interest [Kellison (1970)], $s_{\overline{k}|r}^{(n)}$ stands for the accumulated value, at rate of interest r, of payments of amounts $1/n$ invested at times $1/n, 2/n, \ldots, k$ (k a multiple of $1/n$). Clearly

$$s_{\overline{k}|r}^{(n)} = \sum_{j=1}^{nk} (1+r)^{k-j/n}/n$$
$$= \left[(1+r)^k - 1\right]/n\left[(1+r)^{1/n} - 1\right].$$

When $n \to \infty$, that is to say when payments are made 'continuously', this converges to

$$\bar{s}_{\overline{k}|\gamma} = (\mathrm{e}^{\gamma k} - 1)/\gamma,$$

where $\gamma = \log(1+r)$. Define a random counterpart of $s_{\overline{k}|}^{(n)}$ as

$$S_{nk} = \sum_{i=1}^{nk} n^{-1} \prod_{j=i+1}^{nk} (1 + R_{nj}) \tag{1}$$

where $\{R_{nj}, j \geq 1\}$ is a sequence of i.i.d. random variables. R_{nj} is the rate of return earned over the period $((j-1)/n, j/n)$. Under suitable conditions on $\{R_n, n \geq 1\}$ it will be seen that $\{S_n, n \geq 1\}$ converges weakly to a diffusion.

2.2. Now turn to discounted values. $a_{\overline{k}|}^{(n)}$ stands for the value of the same payments of $1/n$ unit invested at times $1/n, 2/n, \ldots, k$, discounted to time 0.

$$a_{\overline{k}|r}^{(n)} = \sum_{j=1}^{nk} (1+r)^{-j/n}/n$$
$$= \left[1 - (1+r)^{-k}\right]/n\left[(1+r)^{1/n} - 1\right].$$

Its limit as $n \to \infty$ is

$$\bar{a}_{\overline{k}|\gamma} = (1 - \mathrm{e}^{-\gamma k})/\gamma.$$

In accordance with equation (1), we define the random counterpart of $a_{\overline{k}|}^{(n)}$ as

$$\mathscr{A}_{nk} = \sum_{i=1}^{nk} n^{-1} \prod_{j=1}^{i} (1 + R_{nj})^{-1}.$$

Notice that for all n and k

$$\mathscr{A}_{nk} = U_{nk}^{-1} S_{nk}$$

where

$$U_{nk} = \prod_{i=1}^{nk} (1 + R_{ni}) \tag{2}$$

is the accumulated value at time k of one unit invested at time 0. It will be seen that $\{\mathscr{A}_n, n \geq 1\}$ has a weak limit (under appropriate conditions on $\{R_n, n \geq 1\}$), but that it is not a diffusion.

2.3. The determination of the limits of $\{S_n\}$ and $\{\mathscr{A}_n\}$ is made complicated by their being sums of products. The situation is a lot simpler with $\{U_n\}$, however [see equation (2)]. If we define

$$R_{nj} = \exp \gamma_{nj} - 1$$
$$\gamma_{nj} = n^{-1}\mathrm{E}\gamma_{1j} + n^{-1/2}(\gamma_{1j} - \mathrm{E}\gamma_{1j}). \tag{3}$$

we get

$$U_{nk} = \exp\left(\sum_{j=1}^{nk} \gamma_{nj}\right)$$
$$\xrightarrow{\mathrm{D}} \exp\{\Gamma_k\}, \quad \Gamma_k \sim N(k\mathrm{E}\gamma_{11}, k \operatorname{Var} \gamma_{11})$$

by the Central Limit Theorem. Going one step further, define the process $U_{ns} = U_{nk}$ for $k \leq s < k + 1/n$. The weak convergence of $\{U_{ns}, s \geq 0\}$ is then a consequence of Donsker's Theorem [Billingsley (1968, p. 137)]:

$$\{U_{ns}, s \geq 0\}$$
$$\xrightarrow{\mathrm{D}} \{\exp[s\mathrm{E}\gamma_{11} + (\operatorname{Var} \gamma_{11})W_s], s \geq 0\} \tag{4}$$

where W is Brownian motion.

The weak convergence of $\{\mathscr{A}_n\}$ and $\{S_n\}$ will ultimately rest on (4).

3. Defining $\{R_{nj}, j \geq 1\}$

3.1. Given a particular i.i.d. sequence $\{R_{1j}, j \geq 1\}$, it is required to define the sequence of $n$thly rates of return $\{R_{nj}, j \geq 1\}$. One way of doing so has already been pointed out in Section 2. Suppose $P(R_{11} > -1) = 1$, $\operatorname{Var} \log(1 + R_{11}) < \infty$, and set

$$\gamma_{11} = \log(1 + R_{11}),$$
$$\gamma_{nj} \triangleq n^{-1}\mathrm{E}\gamma_{11} + n^{-1/2}(\gamma_{11} - \mathrm{E}\gamma_{11}), \tag{5}$$
$$R_{nj} = \exp \gamma_{nj} - 1. \tag{6}$$

('$\triangleq$' means 'equal in distribution'.) Finally, suppose $\{R_{nj},\ j \geq 1\}$ is independent.

An intuitive justification of (5) is that it leaves annual logarithmic growth rates unchanged as far as means and variances are concerned,

$$\mathrm{E} \sum_{j=1}^{n} \gamma_{nj} \equiv \mathrm{E}\gamma_{11},$$

$$\mathrm{Var} \sum_{j=1}^{n} \gamma_{nj} \equiv \mathrm{Var}\ \gamma_{11}.$$

3.2. Another way of proceeding is to arrange for the means and variances of the annual arithmetic growth rates to be constant for all n. This can be achieved by setting

$$R_{n1} \triangleq r_n + q_n(R_{11} - \mathrm{E}R_{11})/(\mathrm{Var}\ R_{11})^{1/2} \tag{7}$$

where

$$r_n = (1 + \mathrm{E}R_{11})^{1/n} - 1,$$

$$q_n^2 = \left[\mathrm{E}(1 + R_{11})^2\right]^{1/n} - (1 + \mathrm{E}R_{11})^{2/n}.$$

These equations ensure that

$$\mathrm{E} \prod_{j=1}^{n} (1 + R_{nj}) \equiv \mathrm{E}(1 + R_{11}),$$

$$\mathrm{E} \prod_{j=1}^{n} (1 + R_{nj})^2 \equiv \mathrm{E}(1 + R_{11})^2.$$

[This assumes $\mathrm{Var}\ R_{11} < \infty$ and $P(R_{11} > -1) = 1$.]

4. Weak convergence of $\{\mathscr{A}_n\}$ and $\{S_n\}$

4.1. Here are a few definitions and theorems from the theory of weak convergence of probability measures. For more details refer to Billingsley (1968) and D. Pollard (1984).

Definitions

(a) $D[0, T] = \{x\colon [0, T] \to \mathbb{R} \mid x_{t+} = x_t\ \forall 0 \leq t < T,\ x_{t-} \text{ exists } \forall 0 < t \leq T\}$,
$D[0, \infty) = \{x\colon [0, \infty) \to \mathbb{R} \mid x_{t+} = x_t\ \forall t \geq 0,\ x_{t-} \text{ exists } \forall t > 0\}$.
The elements of $D[0, T]$ (or $D[0, \infty)$) are called cadlag functions.

(b) $C[0, T] = \{x\colon [0, T] \to \mathbb{R} \mid x_t \text{ is continuous at all } 0 \leq t \leq T\}$,
$C[0, \infty) = \{x\colon[0, \infty) \to \mathbb{R} \mid x_t \text{ is continuous at all } t \geq 0\}$,

(c) $x_n, x \in D[0, T]$. We write $x_n \xrightarrow{J_T} x$ if $d_T(x_n, x) \to 0$, $d_T(\cdot,\cdot)$ being the Skorohod metric on $D[0, T]$ [see Billingsley (1968, pp. 112–113)]. If X_n, X are random elements of $D[0, T]$, $X_n \xrightarrow{J_T} X$ will mean weak convergence of X_n to X when $D[0, T]$ is equipped with its Skorohod topology and Borel σ-field.

(d) $x_n,\ x \in D[0, T]$. We write $x_n \xrightarrow{U_T} x$ if $\| x_n - x \|_T \to 0$, with
$$\| x_n - x \|_T = \sup_{0 \leq s \leq T} | x_n(s) - x(s) |.$$
If X_n, X are random elements of $D[0, T]$, $X_n \xrightarrow{U_T} X$ will mean weak convergence of X_n to X when $D[0, T]$ is equipped with its uniform topology and projection σ-field [see chapter 5 of Pollard (1984)].

(e) $x_n,\ x \in D[0, \infty)$. Let
$$d(x_n, x) = \sum_{k=1}^{\infty} 2^{-k} \min(1, \| x_n - x \|_k).$$
Convergence with respect to this distance will be written $x_n \xrightarrow{U_\infty} x$. Weak convergence $X_n \xrightarrow{U_\infty} X$ is defined as in (d).

Theorem 1. *$X_n \xrightarrow{J_T} X$, $P(X \in C[0, T]) = 1$ implies $X_n \xrightarrow{U_T} X$.*

Theorem 2. *Suppose $P(X \in C[0, \infty)) = 1$. Then $X_n \xrightarrow{U_\infty} X$ if and only if $X_n \xrightarrow{U_T} X$ for all $T = 1, 2, \ldots$*

Theorem 3 (*Representation Theorem*). *$X_n \xrightarrow{U_T} X$, $P(X \in C[0, T]) = 1$ imply that there exist $\{\tilde{X}_n,\ n \geq 1\}$, $\tilde{X}$ defined on the same probability space such that $\tilde{X}_n \triangleq X_n$, $\tilde{X} \triangleq X$ and*

$$\tilde{X}_n(\omega) \xrightarrow{U_T} \tilde{X}(\omega) \text{ for almost all } \omega.$$

4.2. For each $n \geq 1$, consider streams of payments $\{p_{nk},\ k \geq 1\}$, p_{nk} being the payment made at time k/n (in Section 2 we had $p_{nk} \equiv 1/n$). Also, suppose constant initial payments Q_n with

$Q_n \to Q \in \mathbb{R}$. Let $[y]$ denote the largest integer smaller than or equal to y, and define

$$F_{nt} = \sum_{k<[nt]} p_{nk},$$

$$U_{nt} = \prod_{j=1}^{[nt]} (1+R_{nj}) \quad t \geq 1/n,$$

$$= 1 \qquad t < 1/n,$$

$$V_{nt} = 1/U_{nt},$$

$$\mathscr{A}_{nt} = Q_n + \sum_{i=1}^{[nt]} p_{ni} \prod_{j=1}^{i} (1+R_{nj})^{-1}$$

$$= Q_n + \int_0^t V_{ns}\, \mathrm{d}F_{ns},$$

$$S_{nt} = U_{nt}\mathscr{A}_{nt}$$

$$= Q_n \prod_{j=1}^{[nt]} (1+R_{nj}) + \sum_{i=1}^{[nt]} p_{ni} \prod_{j=i+1}^{[nt]} (1+R_{nj})$$

$$= Q_n U_{nt} + \int_0^t U_{nt}/U_{ns}\, \mathrm{d}F_{ns}.$$

Lemma 1. *$U_n \xrightarrow{U_\infty} U = \mathrm{e}^{\overline{W}}$, $V_n \xrightarrow{U_\infty} V = \mathrm{e}^{-\overline{W}}$ where*

$$\overline{W}_t = \gamma t + \sigma W_t,$$

W_t being Brownian motion. Furthermore,

(a) if $\{R_n\}$ is defined by equation (6), then

$$\gamma = \gamma_a = \mathrm{E}\gamma_{11}, \quad \sigma^2 = \sigma_a^2 = \mathrm{Var}\ \gamma_{11}; \tag{8}$$

(b) if $\{R_n\}$ is defined by equation (7), then

$$\gamma = \gamma_b = \log(1+\mathrm{E}R_{11}) - \sigma_b^2/2,$$
$$\sigma^2 = \sigma_b^2 = \log\left[\mathrm{E}(1+R_{11})^2/(1+\mathrm{E}R_{11})^2\right]. \tag{9}$$

Proof. (a) $\log U_n \xrightarrow{J_T} \overline{W}$ by Donsker's Theorem. This implies $\log U_n \xrightarrow{U_T} \overline{W}$ by Theorem 1, and then Theorem 2 gives $\log U_n \xrightarrow{U_\infty} \overline{W}$. By the continuous Mapping Theorem [Pollard (1984, p. 70)] $U_n \xrightarrow{U_\infty} \mathrm{e}^{\overline{W}}$ and $V_n \xrightarrow{U_\infty} \mathrm{e}^{-\overline{W}}$.

(b) I will show that for each $T \in \mathbb{N}$ U_n converges weakly (J_T) to the solution of the stochastic differential equation

$$\mathrm{d}\xi_t = r\xi_t\, \mathrm{d}t + \sigma\xi_t\, \mathrm{d}W_t, \quad \xi_0 = 1, \tag{10}$$

where $r = \log(1+\mathrm{E}R_{11})$ and σ is as in (9). This implies $\xi_t = \exp \overline{W}_t$. The proof can then be finished just as in (a).

Refer to pp. 184–208 of Gihman and Skorohod (1979), designated by GS in the sequel.

Let $T=1$ for simplicity; the same arguments work for any T. Let $m_n = n$, $t_{nk} = k/n$ and

$$\xi_{nk} = \prod_{j=1}^{nk} (1+R_{nj}) = U_{nk/n},$$

$$\xi_n(t) = U_{nt}.$$

Then

$$\xi_{nk+1} = (1+R_{nk+1})\xi_{nk}$$
$$= \xi_{nk} + \alpha_{nk}\Delta t_{nk} + \beta_{nk}\Delta\psi_{nk}$$

where

$$\alpha_{nk} = nr_n\xi_{nk}, \quad \beta_{nk} = (n\ \mathrm{Var}\ R_{n1})^{1/2}\xi_{nk},$$

$$\psi_{nk} = (n\ \mathrm{Var}\ R_{n1})^{-1/2} \sum_{j=1}^{k} (R_{nj} - r_n).$$

Clearly (GS, p. 188)

$$\alpha_{nk} = a_{nk}(\xi_{nk}), \quad a_{nk}(x) = nr_n x,$$

$$\beta_{nk} = b_{nk}(\xi_{nk}), \quad b_{nk}(x) = (n\ \mathrm{Var}\ R_{n1})^{1/2} x.$$

Now for any $a, b > 0$

$$n(a^{1/n} - b^{1/n}) \to \log a/b.$$

Therefore

$$a_n(x) \to rx, \quad b_n(x) \to \sigma x$$

with r and σ as in (10). Furthermore $\psi_n \xrightarrow{J_1} W$ by Donsker's Theorem. Thus the conditions of Theorem 2 and 13 of GS (pp. 190 and 208) are all satisfied, i.e. $\{P_n\xi_n^{-1},\ n \geq 1\}$ is weakly compact, with finite-dimensional distributions converging to those of the unique solution of (10). This implies $U_n \xrightarrow{J_1} \mathrm{e}^{\overline{W}}$. $\square$

Remark 4.1. Equations (8) and (9) show that γ and σ depend on the way $\{R_n\}$ is defined (given the same initial distribution for R_{11}). In order to assess how different (γ_a, σ_a^2) and (γ_b, σ_b^2) may be, consider the cumulant generating function of $\gamma_{11} = \log(1+R_{11})$,

$$K(t) = \log \mathrm{E}\exp(t\gamma_{11}).$$

If it exists in a neighborhood of $t=0$, $K(t)$ has the expansion

$$K(t) = \sum_{j\geq 1} t^j k_j/j! \tag{11}$$

where

$$k_1 = \mathrm{E}\gamma_{11}, \quad k_2 = \mathrm{Var}\ \gamma_{11}, \quad k_3 = \mathrm{E}(\gamma_{11} - \mathrm{E}\gamma_{11})^3,$$
$$k_4 = \mathrm{E}(\gamma_{11} - \mathrm{E}\gamma_{11})^4 - 3(\mathrm{Var}\ \gamma_{11})^2,$$

etc. [Cramer (1946, pp. 185–187)]. From equations (8) and (9),

$$\gamma_a = k_1, \quad \sigma_a^2 = k_2$$

and

$$\sigma_b^2 = K(2) - 2K(1),$$
$$\gamma_b = K(1) - [K(2) - 2K(1)]/2.$$

Equation (11) tells us that

$$\sigma_b^2 = k_2 + k_3 + (7/12)k_4 + \dots$$
$$\gamma_b = k_1 - k_3/3 - k_4/4 + \dots$$

We conclude that if the cumulants of γ_{11} of third and higher order are negligible, then (γ_a, σ_a^2) and (γ_b, σ_b^2) will be close.

In the special case where $\gamma_{11} \sim \mathrm{N}(m, s^2)$ (which is equivalent to saying that $1 + R_{11}$ is lognormal) we see that $(\gamma_a, \sigma_a^2) = (\gamma_b, \sigma_b^2) = (m, s^2)$. Either way the normal distribution reproduces itself. □

Remark 4.2. Cox et al. (1979, pp. 246–255) use an artifice similar to (5) in order to show that their discrete-time option formula converges to the one derived by Black and Scholes (1973). Their initial rate of return γ_{11} takes only two values

$$\gamma_{11} = g_1 \quad \text{with prob. } q,$$
$$\quad = g_2 \quad \text{with prob. } 1 - q$$

($g_1 = \log u$, $g_2 = \log d$ in their notation). They define i.i.d. random variables $\{\gamma_{nk}, 1 \le k \le n\}$ with

$$\gamma_{n1} = n^{-1/2}(\mathrm{Var}\ \gamma_{11})^{1/2}$$
$$\quad \text{with prob. } q_n = 1/2 + \mathrm{E}\gamma_{11}/2(n\ \mathrm{Var}\ \gamma_{11})^{1/2}$$
$$\quad = -n^{-1/2}(\mathrm{Var}\ \gamma_{11})^{1/2} \quad \text{with prob. } 1 - q_n.$$

Hence

$$\mathrm{E}\sum_{k=1}^{n} \gamma_{nk} = \mathrm{E}\gamma_{11},$$

$$\mathrm{Var} \sum_{k=1}^{n} \gamma_{nk} = s_n^2 = \mathrm{Var}\ \gamma_{11} - (\mathrm{E}\gamma_{11})^2/n$$
$$\quad \to \mathrm{Var}\ \gamma_{11} \quad \text{as } n \to \infty.$$

This asymptotically preserves the mean and variance of the annual logarithmic growth rate. Lindeberg's condition

$$\forall \epsilon > 0, \quad n\mathrm{E}(\gamma_{n1} - \mathrm{E}\gamma_{n1})^2 1_{\{|\gamma_{n1} - \mathrm{E}\gamma_{n1}| > \epsilon s_n\}} \to 0$$

is satisfied and thus

$$\left\{ \overline{W}_{nt} = \sum_{k=1}^{[nt]} \gamma_{nk} \right\}$$
$$\xrightarrow{J_1} \left\{ \overline{W}_t = \mathrm{E}\gamma_{11} t + (\mathrm{Var}\ \gamma_{11})^{1/2} W_t \right\}.$$

This is consistent with the convergence of the discrete pricing formula to the continuous one, which is directly based on $\overline{W}_t$. □

4.3. Now suppose that the payment measures $\{F_n\}$ converge to a function F in the following sense:

$$\lim_{n\to\infty} \| F_n - F \|_T = 0 \qquad \forall T < \infty \qquad (12a)$$
$$\sup_n V_T F_n < \infty \qquad (12b)$$

where $V_T F_n$ is the total variation of F_n over $[0, T]$. Conditions (12) ensure that F is left-continuous with $V_T F \le \sup_n V_T F_n$ for each $T < \infty$.

The following lemma is an adaptation of the classical Helly Convergence Theorem [Kolmogorov and Fomin (1970, p. 370)].

Lemma 2. *Let $x \in D[0, \infty)$ and $\{F_n\}$ satisfy (12). Then*

$$\sup_{0\le t\le T} \left| \int_0^t x_s\, \mathrm{d}(F_{ns} - F_s) \right| \to 0 \text{ as } n \to \infty$$

for all $T < \infty$.

Proof. Let $\phi \in D[0, T]$ be a step function such that

$$\| \phi - x \|_T < \epsilon/4 \sup_n V_T F_n.$$

Then

$$\left| \int_0^t x_s\, \mathrm{d}(F_{ns} - F_s) \right| \le \left| \int_0^t (x_s - \phi_s)\, \mathrm{d}(F_{ns} - F_s) \right|$$
$$+ \left| \int_0^t \phi_s\, \mathrm{d}(F_{ns} - F_s) \right|.$$

For any $t \le T$, the first term on the right is smaller than

$$\| \phi - x \|_T V_T(F_n - F) \le 2 \| \phi - x \|_T \sup_n V_T F_n$$
$$< \epsilon/2.$$

The second term is equal to

$$\left|\phi_s(F_{ns}-F_s)\right|_{s=0}^{s=t}-\int_0^t(F_{ns}-F_s)\,\mathrm{d}\phi_s\Big|$$
$$\le(2\,\|\phi\|_T+V_T\phi)\,\|F_n-F\|_T$$

for $t\le T$. This is smaller than $\epsilon/2$ for all n large enough. □

Lemma 3. *Suppose $\{F_n\}$ satisfies (12). Let X_n, X be random elements of $D[0,\infty)$ such that $X_n\xrightarrow{U_\infty}X$ with $P(X\in C[0,\infty))=1$, and*

$$Y_{nt}=\int_0^t X_{ns}\,\mathrm{d}F_{ns},$$
$$Y_t=\int_0^t X_s\,\mathrm{d}F_s.$$

Then $Y_n\xrightarrow{U_\infty}Y$.

Proof. Use the Representation Theorem to obtain $\tilde X_n\triangleq X_n$, $\tilde X\triangleq X$ such that $\tilde X_n(\omega)\xrightarrow{U_T}\tilde X(\omega)$ for almost all ω. Then

$$\left\{\tilde Y_{nt}=\int_0^t\tilde X_{ns}\,\mathrm{d}F_{ns},\ t\ge0\right\}\triangleq Y_n,$$
$$\left\{\tilde Y_t=\int_0^t\tilde X_s\,\mathrm{d}F_s,\ t\ge0\right\}\triangleq Y,$$

and

$$|\tilde Y_{nt}-\tilde Y_t|$$
$$\le\left|\int_0^t(\tilde X_{ns}-\tilde X_s)\,\mathrm{d}F_{ns}\right|+\left|\int_0^t\tilde X_s\,\mathrm{d}(F_{ns}-F_s)\right|$$
$$\le\|\tilde X_n-\tilde X\|_T\,\sup_n V_TF_n$$
$$+\sup_{0\le t\le T}\left|\int_0^t\tilde X_s\,\mathrm{d}(F_{ns}-F_s)\right|$$
$$\to0\text{ almost surely}$$

by Lemma 2. This implies $Y_n\xrightarrow{U_T}Y$ for all $T<\infty$, and so $Y_n\xrightarrow{U_\infty}Y$. □

Proposition 1. *Suppose $\{F_n\}$ satisfies (12) with*

$$F_t=\int_0^t p_s\,\mathrm{d}s \tag{13}$$

where p_s is a measurable function that is bounded on bounded intervals. Then

(a) $\mathscr{A}_n\xrightarrow{U_\infty}\mathscr{A}$ where

$$\mathscr{A}_t=Q+\int_0^t p_sV_s\,\mathrm{d}s, \tag{14}$$

(b) $S_n\xrightarrow{U_\infty}S$, S being the unique solution of the stochastic differential equation

$$\mathrm{d}S_t=\left[(\gamma+\sigma^2/2)S_t+p_t\right]\mathrm{d}t+\sigma S_t\,\mathrm{d}W_t, \tag{15}$$
$$S_0=Q\quad\text{a.s.}$$

Proof. Part (a) is a direct consequence of Lemmas 1 and 3. Next

$$S_n=U_n\mathscr{A}_n\xrightarrow{U_\infty}U\mathscr{A}=\left\{Q\,\mathrm{e}^{\overline{W}_t}+\mathrm{e}^{\overline{W}_t}\mathscr{A}_t,\ t\ge0\right\}=S.$$

Equation (15) follows from the product rule for Itô differentials [Gihman and Skorohod (1972, p. 22)]. Its solution is unique by the classical existence and uniqueness theorems [e.g. Gihman and Skorohod (1972, p. 40)] □

Remark 4.3. Conditions (12) will always be satisfied when F_n is a left continuous step approximation of a continuous F with bounded variation,

$$F_{nt}=F(k/n),\quad(k-1)/n<t\le k/n,$$
$$k=1,2,\ldots$$

If F is differentiable [equation (13)], its derivative p does not have to be continuous. □

Remark 4.4. As the rest of the paper deals mostly with the moments of $\mathscr{A}$ and S, it is of some interest to ask whether the moments of $\mathscr{A}_n$ and S_n converge to those of their weak limits.

First, consider the first moments of discounted values. Condition (12b) ensures that

$$\mathrm{E}\,|\mathscr{A}_t|\le|Q|+(V_tF)\sup_{0\le s\le t}\mathrm{E}V_s<\infty$$

for any finite t. However, $\mathrm{E}\mathscr{A}_{nt}$ does not exist if

$$\mathrm{E}\exp-\gamma_{n1}=\infty,\quad\gamma_{n1}=\log(1+R_{n1}).$$

[This may happen with the first way of defining R_{nj}, see equation (6)]. Consequently, if

$$\mathrm{E}\exp-c\gamma_{11}=\infty$$

for all $c>0$, then $\mathrm{E}\mathscr{A}_{nt}$ cannot exist for any n and $t\ge1/n$.

Suppose there exists $c>0$ such that $\mathrm{E}\exp-c\gamma_{11}<\infty$. Then both $\mathrm{E}V_{ns}$ and $\mathrm{E}V_{nr}V_{ns}$ are finite for all n greater than some n_0 (depending on c). Furthermore

$$\sup_{n_0}\ \sup_{0\le r,s\le t}\ \mathrm{E}V_{nr}V_{ns}<\infty$$

and thus

$$\sup_{n_0}\mathrm{E}\mathscr{A}_{nt}^2<\infty.$$

This means that $\{\mathscr{A}_{nt}, n \geq n_0\}$ is uniformly integrable. Since $\mathscr{A}_{nt}$ converges weakly to $\mathscr{A}_t$, we conclude that $\mathrm{E}\mathscr{A}_{nt}$ converges to $\mathrm{E}\mathscr{A}_t$ [Billingsley (1968, p. 32)].

The same argument works for higher moments: if $\mathrm{E} \exp -c\gamma_{11} < \infty$ for some $c > 0$, then

$$\mathrm{E}\mathscr{A}_{nt_1} \ldots \mathscr{A}_{nt_k} \to \mathrm{E}\mathscr{A}_{t_1} \ldots \mathscr{A}_{t_k} \in \mathbb{R}$$

as $n \to \infty$, for any finite set $(t_1, \ldots, t_k)$.

An identical result holds for accumulated values: if $\mathrm{E} \exp c\gamma_{11} < \infty$ for some $c > 0$, then

$$\mathrm{E}S_{nt_1} \ldots S_{nt_k} \to \mathrm{E}S_{t_1} \ldots S_{t_k} \in \mathbb{R}$$

for any finite set $(t_1, \ldots, t_k)$.

(I am indebted to one of the referees for raising this question.) □

5. Moments of U, V, S and $\mathscr{A}$

5.1. Consider now the moments of

$$U_t = \exp \overline{W}_t, \quad V_t = \exp - \overline{W}_t,$$

$$\mathscr{A}_t = Q + \int_0^t p_s V_s \,\mathrm{d}s,$$

$$S_t = QU_t + \int_0^t p_s U_t / U_s \,\mathrm{d}s.$$

Let

$$\alpha_k = k\gamma + k^2\sigma^2/2, \quad \delta_k = k\gamma - k^2\sigma^2/2$$

for $k \geq 0$. Clearly

$$\mathrm{E}U_t^k = \exp(\alpha_k t), \quad \mathrm{E}V_t^k = \exp(-\delta_k t) \tag{16}$$

for all $k \geq 0$.

S_t has finite moments of all orders because $S_0 = Q$ is a constant [see the exponential bound for $\mathrm{E}S_t^{2m}$ on p. 48 of Gihman and Skorohod (1972)]. This also extends to $\mathscr{A}_t = V_t S_t$. Both S and $\mathscr{A}$ are continuous w.p.1, and are thus progressively measurable. These facts imply that it is always permitted to interchange expectations and integrals over finite intervals, e.g.

$$\mathrm{E}\int_\alpha^\beta S_t^k (\text{resp. } \mathscr{A}_t^k) \,\mathrm{d}t = \int_\alpha^\beta \mathrm{E}S_t^k (\text{resp. } \mathrm{E}\mathscr{A}_t^k) \,\mathrm{d}t$$

since $\int_\alpha^\beta |\mathrm{E}S_t^k|$ (resp. $|\mathrm{E}\mathscr{A}_t^k|$) $\mathrm{d}t$ is always finite.

5.2. The moments of S_t are found in the following way. Refer to equation (15). Applying Itô's formula with $f(x) = x^k$ we get

$$\begin{aligned}\mathrm{d}S_t^k &= kS_t^{k-1}\,\mathrm{d}S_t + (1/2)f''(S_t)\sigma^2 S_t^2 \,\mathrm{d}t \\ &= \left(\alpha_k S_t^k + kp_t S_t^{k-1}\right)\mathrm{d}t + k\sigma S_t^k \,\mathrm{d}W_t.\end{aligned}$$

Writing this in integral form and then taking expectations,

$$\mathrm{E}S_t^k = Q^k + \int_0^t \left(\alpha_k \mathrm{E}S_u^k + kp_u \mathrm{E}S_u^{k-1}\right)\mathrm{d}u \tag{17}$$

since $\mathrm{E}\int_0^t S_u^k \,\mathrm{d}W_u = 0$. This means

$$\mathrm{dE}S_t^k/\mathrm{d}t = \alpha_k \mathrm{E}S_t^k + kp_t \mathrm{E}S_t^{k-1}. \tag{18}$$

The moments can thus be found recursively.

5.3. In the case of constant payments we have the following result.

Proposition 2. *When $Q = 0$ and $p_t \equiv 1$,*

$$\mathrm{E}S_t^k = k! \sum_{j=0}^k b_{kj} \,\mathrm{e}^{\alpha_j t} \tag{19}$$

for $k = 1, 2, \ldots$, where

$$b_{kj} = \left[\prod_{\substack{i=0\\ i\neq j}}^k (\alpha_j - \alpha_i)\right]^{-1}. \tag{20}$$

Proof. That the right-hand side of equation (19) satisfies (18) is readily verified, observing that $b_{kj}(\alpha_j - \alpha_k) = b_{(k-1)j}$ for $0 \leq j \leq k-1$. It only remains to show that (19) satisfies the initial condition $\mathrm{E}S_0^k = 0$. Define

$$q(x) = \prod_{i=0}^k (x - \alpha_i), \quad f(x) = (y - x)/q(y)$$

where $y \neq \alpha_i$, $0 \leq i \leq k$. Then

$$q'(\alpha_j) = \prod_{\substack{i=0\\ i\neq j}}^k (\alpha_j - \alpha_i) = b_{kj}^{-1}.$$

By Lagrange's interpolation formula [Atkinson (1978, p. 115)],

$$\begin{aligned}f(x) &= \sum_{j=0}^k f(\alpha_j) q(x) / \left[(x - \alpha_j) q'(\alpha_j)\right] \\ &= \sum_{j=0}^k \left[(y - \alpha_j)/q(y)\right] q(x) \\ &\qquad /\left[(x - \alpha_j) q'(\alpha_j)\right].\end{aligned}$$

Letting $x \to y$ yields

$$0 = \sum_{j=0}^k 1/q'(\alpha_j) = \sum_{j=0}^k b_{kj}. \quad \square$$

5.4. Now turn to $\mathscr{A}_t$. From (14) and (16)

$$\mathrm{E}\mathscr{A}_t = Q + \int_0^t p_s \exp(-\delta_1 s)\, \mathrm{d}s.$$

The second moment of $\mathscr{A}_t$ can be obtained by squaring (14) and then taking expectations,

$$\mathrm{E}\mathscr{A}_t^2 = Q^2 + 2Q\int_0^t p_s \mathrm{E}V_s\, \mathrm{d}s + \int_0^t\int_0^t p_s p_u \mathrm{E}V_s V_u\, \mathrm{d}s\, \mathrm{d}u.$$

For arbitrary rates of return processes $\{\gamma_t\}$ this appears to be the only possible line of action [e.g. Panjer and Bellhouse (1980)]. With white noise rates of return, however, the duality between accumulating and discounting provides a much simpler technique for deriving all the moments of $\mathscr{A}_t$.

Lemma 4. *For fixed* $T > 0$

$$\mathscr{A}_t - Q = \int_0^T p_s \exp(-\overline{W}_s)\, \mathrm{d}s \triangleq \int_0^T p_{T-s} \exp(-\overline{W}_T + \overline{W}_s)\, \mathrm{d}s.$$

Proof. The processes

$$X = \{W_{T-s}, 0 \le s \le T\}$$

and

and $\quad Y = \{W_T - W_s, 0 \le s \le T\}$

are Gaussian with the same mean and covariance functions. Their finite-dimensional distributions are thus identical [Doob (1953, p. 72)]. We infer that their distributions (as random elements of $D[0, T]$) are identical [Billingsley (1968, p. 123)]. Define h: $D[0, T] \to \mathbb{R}$ as

$$h(x) = \int_0^T p_{T-s} \exp[-\gamma(T-s) - \sigma x_s]\, \mathrm{d}s.$$

Then $h(X) \triangleq h(Y)$, that is to say

$$\begin{aligned}\int_0^T p_s \exp(-\overline{W}_s)\, \mathrm{d}s &= \int_0^T p_{T-s} \exp[-\gamma(T-s) - \sigma W_{T-s}]\, \mathrm{d}s \\ &\triangleq \int_0^T p_{T-s} \exp[-\gamma(T-s) - \sigma W_T + \sigma W_s]\, \mathrm{d}s \\ &= \int_0^T p_{T-s} \exp[-\overline{W}_T + \overline{W}_s]\, \mathrm{d}s. \quad \square\end{aligned}$$

Define

$$B_t = \exp(-\overline{W}_t)\int_0^t \exp(\overline{W}_s) p_{T-s}\, \mathrm{d}s, \quad 0 \le t \le T. \tag{21}$$

Compare B_t with the solution of equation (15) (when $S_0 = 0$),

$$S_t = \exp(\overline{W}_t)\int_0^t \exp(-\overline{W}_s) p_s\, \mathrm{d}s. \tag{22}$$

It is seen that B_t represents the accumulated value, at rates of return $-\mathrm{d}\overline{W}_t/\mathrm{d}t = -\gamma - \sigma\, \mathrm{d}W_t/\mathrm{d}t$, of the payments p, taken in reverse order starting at time T. To this extent, accumulating and discounting are dual operations, as they are when rates of return are constant (this will be discussed further in Section 7.1).

In order to find $\mathrm{E}(\mathscr{A}_T - Q)^k = \mathrm{E}B_T^k$, one can therefore solve

$$\mathrm{d}EB_t^k/\mathrm{d}t = -\delta_k \mathrm{E}B_t^k + kp_{T-t}\mathrm{E}B_t^{k-1}, \quad 0 \le t \le T,$$

with $\mathrm{E}B_0^k = 0$ for $k \ge 1$. This is equation (18) with γ replaced with $-\gamma$ and p_t with p_{T-t}.

In particular, when $Q = 0$ and $p_t \equiv 1$,

$$\mathrm{E}\mathscr{A}_t^k = k!\sum_{j=0}^k c_{kj}\, \mathrm{e}^{-\delta_j t} \tag{23}$$

with

$$c_{kj} = \left[\prod_{\substack{i=0\\ i\ne j}}^k (\delta_i - \delta_j)\right]^{-1}. \tag{24}$$

These are equations (19) and (20) with $-\delta_j$ substituted for α_j.

5.5. $\mathrm{Cov}(S_t, S_{t+v})$ and $\mathrm{Cov}(\mathscr{A}_t, \mathscr{A}_{t+v})$ will now be derived. Fix t and define

$$g(v) = \mathrm{Cov}(S_t, S_{t+v}), \quad v \ge 0.$$

Write (15) in integral form and substract (17) (with $k = 1$) to obtain

$$S_t - \mathrm{E}S_t = \int_0^t \alpha_1(S_u - \mathrm{E}S_u)\, \mathrm{d}u + \int_0^t \sigma S_u\, \mathrm{d}W_u. \tag{25}$$

Then

$$\begin{aligned} g(v) &= g(0) + \mathrm{E}(S_t - \mathrm{E}S_t) \times [(S_{t+v} - \mathrm{E}S_{t+v}) - (S_t - \mathrm{E}S_t)] \\ &= g(0) + \mathrm{E}(S_t - \mathrm{E}S_t) \times \left[\int_t^{t+v} \alpha_1(S_u - ES_u)\, \mathrm{d}u + \int_t^{t+v} \sigma S_u\, \mathrm{d}W_u\right] \\ &\qquad \text{(from (25))} \\ &= g(0) + \int_0^v \alpha_1 g(u)\, \mathrm{d}u \end{aligned} \tag{26}$$