Insurance: Mathematics and Economics 8 (1989) 71–76
North-Holland

Stability of pension systems when rates of return are random *

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Consider a funded pension plan, and suppose actuarial gains or losses are amortized over a fixed number of years. The paper aims at assessing how contributions (C) and fund levels (F) are affected when the rates of return of the plans's assets form an i.i.d. sequence of random variables. This is achieved by calculating the mean and variance of C_t and F_t for $t \leqslant \infty$.

Keywords: Pension funding, Random rates of return, Actuarial gains and losses.

1. Introduction

This paper examines the effect of random rates of return on pension fund levels and contributions. The funding methods considered are those which

(1) determine an actuarial liability (AL) and a normal cost (NC) at every valuation date; and
(2) amortize individual inter-valuation gains or losses over a fixed number of years (e.g. 5 or 15).

These methods have been used by actuaries in Canada and the United States.

Remarks. 1. A similar study has been done of funding methods which satisfy (1) but adjust the normal cost by a constant fraction of the actuarial liability. See Dufresne (1986a, 1988).
2. Pension mathematics and gain and loss analysis are described in Trowbridge (1952), Winklevoss (1977) and Lynch (1979).

2. Notation

AL = actuarial liability,
B = benefit outgo,
C = overall contribution,
F = fund level,
i = valuation rate of interest,
L_t = actuarial loss during $(t-1, t)$,
m = amortization period for actuarial losses,
NC = normal cost,
P = adjustment of normal cost,
r = mean rate of return on assets,
r_t = rate of return on assets during $(t-1, t)$,
UL_t = unfunded liability at time t,
UL_t^A = unfunded liability at time t if all actuarial assumptions work out during $(t-1, t)$,
β_k $= a_{\overline{m-k}|}/\ddot{a}_{\overline{m}|}$, $1 \leqslant k \leqslant m-1$,
λ_k $= \ddot{a}_{\overline{m-k}|}/\ddot{a}_{\overline{m}|}$, $0 \leqslant k \leqslant m-1$,
σ^2 = variance of r_t.

3. Description of model and basic equations

In order to isolate the effect of fluctuating rates of return (and keep the model tractable), the following assumptions are made.

I. The population is stationary from the start.
II. Except for rates of return, all actuarial assumptions are consistently realized.
III. There is no inflation on benefits.
IV. The rates of return $\{r_t, t \geqslant 1\}$ form an i.i.d. sequence with mean r and variance σ^2. r_t is the rate earned on assets during the period $(t-1, t)$.

Suppose the pension plan is set up at time $t=0$. Given the assumptions above, the assets process satisfies

$$F_t = (1+r_t)(F_{t-1} + C_{t-1} - B), \quad t \geqslant 1, \tag{1}$$

where F is the fund level, C the contribution and B the benefit outgo. B is constant from assumptions I to III. On the liabilities side we have

$$AL = (1+i)(AL + NC - B), \tag{2}$$

where i is the valuation rate of interest. This equation is known as the equation of equilibrium.

* This research was supported by the Natural Sciences and Engineering Research Council of Canada.

Now define the unfunded liability at time t as $UL_t = AL - F_t$, $t \geq 0$, and the (actuarial) loss experienced during the period $(t-1, t)$ as

$$L_t = UL_t - [\text{value of } UL_t \text{ had all actuarial assumptions been realized during } (t-1, t)].$$

$$= UL_t - UL_t^{A}, \quad t \geq 1. \tag{3}$$

For the time being let $L_t = 0$ for $t \leq 0$. Letting $r_t = i$ in equation (1), and subtracting it from equation (2), we get

$$UL_t^{A} = (1+i)(UL_{t-1} + NC - C_{t-1}). \tag{4}$$

Under the funding methods considered, the contribution at time t is

$$C_t = NC + P_t, \tag{5}$$

$$P_t = \sum_{k=0}^{m-1} L_{t-k}/\ddot{a}_{\overline{m}|}. \tag{6}$$

Here m (an integer) is the amortization period and

$$\ddot{a}_{\overline{m}|} = [1-(1+i)^{-m}]/[1-(1+i)^{-1}].$$

Each L_s is thus liquidated by m payments of amount $L_s/\ddot{a}_{\overline{m}|}$, made at valuation dates s, $s+1, \ldots, s+m-1$. The fact that $\ddot{a}_{\overline{m}|}$ is calculated at rate i ensures that L_s is in fact cancelled out after the mth payment is made.

Remark. It should be observed that all losses are assumed to be amortized in the same fashion, irrespective of their sign. In practice, it may happen that gains (i.e. negative losses) be written off immediately, in order to reduce the unfunded liability or the current contribution.

As they stand, the above equations do not permit the calculation of the moments of F and C. One way to proceed is as follows:

(i) derive a difference equation involving the L's only;
(ii) calculate the moments of the L's from this equation;
(iii) finally, obtain the moments of F and C from those of the L's.

First, let us express UL_t in terms of the L's. We have

$$\begin{aligned} UL_t &= AL - F_t \\ &= (1+i)(AL + NC - B) \\ &\quad - (1+r_t)(F_{t-1} + NC + P_{t-1} - B) \\ &= (1+r_t)(AL - F_{t-1} - P_{t-1}) \\ &\quad - (r_t - i)(AL + NC - B) \\ &= (1+i)(UL_{t-1} - P_{t-1}) \\ &\quad + (r_t - i)\left(UL_{t-1} - P_{t-1} - (1+i)^{-1}AL\right). \end{aligned} \tag{7}$$

In view of equations (3), (4) and (5), this implies

$$L_t = (r_t - i)\left(UL_{t-1} - P_{t-1} - (1+i)^{-1}AL\right), \quad t \geq 1. \tag{8}$$

Equation (7) can be rewritten as

$$UL_t = (1+i)(UL_{t-1} - P_{t-1}) + L_t \tag{9}$$

or

$$UL_t - (1+i)UL_{t-1} = L_t - (1+i)\sum_{s=t-m}^{t-1} L_s/\ddot{a}_{\overline{m}|}. \tag{10}$$

A particular solution of this difference equation is

$$UL_t^{p} = \sum_{k \geq 0} \lambda_k L_{t-k},$$

where the λ's can be determined by direct substitution into equation (10), yielding

$$\begin{aligned} &\lambda_0 L_t + [\lambda_1 - (1+i)\lambda_0] L_{t-1} \\ &\quad + [\lambda_2 - (1+i)\lambda_1] L_{t-2} + \ldots \\ &\quad = L_t - [(1+i)/\ddot{a}_{\overline{m}|}] L_{t-1} - \ldots \\ &\qquad - [(1+i)/\ddot{a}_{\overline{m}|}] L_{t-m}, \end{aligned}$$

which means

$$\lambda_0 = 1$$
$$\lambda_1 = \ddot{a}_{\overline{m-1}|}/\ddot{a}_{\overline{m}|}$$
$$\lambda_2 = \ddot{a}_{\overline{m-2}|}/\ddot{a}_{\overline{m}|}$$
$$\vdots$$
$$\lambda_{m-1} = \ddot{a}_{\overline{1}|}/\ddot{a}_{\overline{m}|}$$
$$\lambda_k = 0, \; k \geq m.$$

A solution of the homogeneous equation

$$UL_t - (1+i)UL_{t-1} = 0$$

is $c(1+i)^t$, c a constant. The solution of the complete equation (10) is therefore [Brand (1966, p. 368)]

$$UL_t = \sum_{k=0}^{m-1} \lambda_k L_{t-k} + UL_0(1+i)^t.$$

The term $UL_0(1+i)^t$ brings to light the fact that the initial unfunded liability ($UL_0 = AL - F_0$) has not been taken into account so far. It is easy to see that including supplementary payments of amount $UL_0/\ddot{a}_{\overline{n}|}$ at times 0, 1,..., $n-1$ will liquidate UL_0 entirely. For the sake of simplicity, let us assume that $n = m$, so we can define $L_0 = UL_0$ and obtain

$$UL_t = \sum_{k=0}^{m-1} \lambda_k L_{t-k}, \quad t \geqslant 0. \tag{11}$$

Equations (6), (8) and (11) now permit the derivation of a difference equation involving the L's only:

$$L_t = (r_t - i)\left[\sum_{k=0}^{m-1} (\lambda_k - 1/\ddot{a}_{\overline{m}|}) L_{t-1-k} - (1+i)^{-1} AL\right]$$

$$= (r_t - i)\left[\sum_{k=1}^{m-1} \beta_k L_{t-k} - (1+i)^{-1} AL\right], \tag{12}$$

where $\beta_k = \lambda_{k-1} - 1/\ddot{a}_{\overline{m}|} = a_{\overline{m-k}|}/\ddot{a}_{\overline{m}|}$ (clearly $\beta_m = \lambda_{m-1} - 1/\ddot{a}_{\overline{m}|} = 0$).

4. Stability conditions

Definition. A sequence $\{y_t\}$ satisfying

$$y_t + \sum_{j=1}^{n} \alpha_j y_{t-j} = w, \quad t \geqslant 1, \tag{13}$$

will be said to be *stable* if there is a finite value y^* such that $y_t \to y^*$ as $t \to \infty$ for any set of initial conditions $y_0, y_{-1}, \ldots, y_{-n+1}$.

It is well known that a necessary and sufficient condition for this kind of stability is that all the roots of the characteristic equation

$$p(z) = z^n + \sum_{j=1}^{n} \alpha_j z^{n-j} = 0$$

be less than one in modulus.

Proposition 1. *If* $\sum|\alpha_j| < 1$, *then* $\{y_t\}$ *is stable.*

Proof. Suppose there exists $z \in C$ such that $p(z) = 0$ and $|z| \geqslant 1$. Then

$$|z|^n \leqslant \sum_{j=1}^{n} |\alpha_j| \, |z|^{n-j} \leqslant |z|^n \sum_{j=1}^{n} |\alpha_j| < |z|^n,$$

a contradiction. □

Proposition 2. *Suppose* $\alpha_j \leqslant 0$ *for all* j. *Then* $\{y_t\}$ *is stable if and only if* $|\Sigma\alpha_j| < 1$.

Proof. Sufficiency is a consequence of Proposition 1. To prove necessity, suppose $|\Sigma\alpha_j| \geqslant 1$, and let $q(z) = z^n p(1/z)$. Then $q(0) = 1$ and

$$q(1) = 1 + \Sigma\alpha_j \leqslant 0.$$

Thus $q(z)$ has at least one root z^* in (0, 1], which implies that $p(z)$ has the root $z^{**} = 1/z^*$ in $[1, \infty)$. □

Remark. That $|\Sigma\alpha_j| \leqslant 1$ is not in general a necessary condition for stability of (13) can be seen by considering the case $i = 0$, $m = 3$ and $r_t - i \equiv p$ in equation (12).

$$L_t - (2p/3) L_{t-1} - (p/3) L_{t-2} = pAL.$$

This sequence is stable for $-3 < p < 1$, while $|\Sigma\alpha_j| = |p|$.

Let us now return to the processes $\{F_t\}$ and $\{C_t\}$.

Definition. A process $\{X_t\}$ will be said to be *pth order stable* if $\{\mathrm{E}X_t^p\}$ is stable.

Since

$$F_t = AL - UL_t$$
$$= AL - \sum_{k=0}^{m-1} \lambda_k L_{t-k},$$
$$C_t = NC + \sum_{k=0}^{m-1} L_{t-k}/\ddot{a}_{\overline{m}|},$$

it turns out that the stability properties of $\{F_t\}$ and $\{C_t\}$ are the same as those of $\{L_t\}$. We will thus consider equation (12), with initial conditions being now arbitrary (imagining that the plan has been in existence for some time before $t = 0$).

First-order stability

We get

$$\mathrm{E}L_t = \mathrm{E}(r_t - i)\left[\sum_{k=1}^{m-1} \beta_k \mathrm{E}L_{t-k} - (1+i)^{-1} AL\right]$$

since r_t is independent of L_{t-k}, $k \geqslant 1$. Applying Proposition 1, we obtain

Proposition 3. *If* $|r-i|\Sigma\beta_k<1$, *then* $\{L_t\}$, $\{F_t\}$ *and* $\{C_t\}$ *are first-order stable, and*

(a) $\lim_{t\to\infty} \mathrm{E}L_t = M_\infty = -(r-i)(1+i)^{-1}AL/(1-(r-i)\Sigma\beta_k)$,

(b) $\lim_{t\to\infty} \mathrm{E}F_t = AL - M_\infty\Sigma\lambda_k$,

(c) $\lim_{t\to\infty} \mathrm{E}C_t = NC + M_\infty m/\ddot{a}_{\overline{m}|}$.

Second-order stability

At this point we make a supplementary assumption:

V. $\mathrm{E}r_t = r = i$.

Using equation (12), this implies

$$\mathrm{E}L_t = 0, \quad t\geq 1,$$

and

$$\mathrm{E}L_tL_s = \mathrm{E}(r_t - i) \times \mathrm{E}\left[\sum_{k=1}^{m-1}\beta_k L_{t-k} - (1+i)^{-1}AL\right]\cdot L_s = 0,$$

for any $t\geq 1$, $s<t$. Thus $\{L_t,\ t\geq 1\}$ becomes a sequence of uncorrelated zero-mean r.v.'s. Equation (12) then gives

$$\operatorname{Var} L_t = \mathrm{E}L_t^2 = \sigma^2\left[\sum_{k=1}^{m-1}\beta_k^2 \operatorname{Var} L_{t-k} + (1+i)^{-2}AL^2\right], \quad t\geq 1,$$
$$= 0, \quad t\leq 0.$$

Using Proposition 2, we get

Proposition 4. $\{L_t\}$, $\{F_t\}$ *and* $\{C_t\}$ *are second-order stable if and only if* $\sigma^2\sum\beta_k^2<1$, *in which case*

$$\lim_{t\to\infty}\operatorname{Var} L_t = V_\infty = \sigma^2(1+i)^{-2}AL^2/\left(1-\sigma^2\sum\beta_k^2\right),$$

$$\lim_{t\to\infty}\operatorname{Var} F_t = V_\infty\sum\lambda_k^2,$$

$$\lim_{t\to\infty}\operatorname{Var} C_t = V_\infty\, m/\left(\ddot{a}_{\overline{m}|}\right)^2.$$

Remarks. 1. The L's are uncorrelated but certainly not independent. For example, let $m=2$, $r=i=0$, $AL=\frac{1}{2}$. Then $\beta_1=\frac{1}{2}$ and

$$L_t = x_t(L_{t-1}-1),$$

where $x_t = r_t/2$. If, furthermore, $L(0)=0$ and $P(x_t=x)=P(x_t=-x)=\frac{1}{2}$, we get

$$P\left(L_t = -x-x^2-\ldots-x^t\right) = P(x_s=x,\ 1\leq s\leq t) = (1/2)^t,$$

$$P(L_1=x)=1/2$$

and

$$P\left(L_1=x,\ L_t=-x-x^2-\ldots-x^t\right)=0.$$

2. Covariances may also be calculated, yielding

$$\operatorname{Cov}(F_t, F_{t+h}) = \sum_{k=0}^{m-h-1}\operatorname{Var}(L_{t-k})\,\ddot{a}_{\overline{m-k}|}\ddot{a}_{\overline{m-k-h}|}/\left(\ddot{a}_{\overline{m}|}\right)^2, \quad 0\leq h<m,$$
$$=0, \quad h\geq m,$$

$$\operatorname{Cov}(C_t, C_{t+h}) = \sum_{k=0}^{m-h-1}\operatorname{Var}(L_{t-k})/\left(\ddot{a}_{\overline{m}|}\right)^2, \quad 0\leq h<m$$
$$=0, \quad h\geq m.$$

5. Numerical illustration

The purpose of this section is to illustrate the results of Proposition 4.

Assumptions

Population	English Life Table No. 13 (males), stationary
Entry age	30 (only)
Retirement age	65
No salary scale, no inflation on salaries	
Benefits	Straight life annuity (2/3 of salary)
Funding method	Entry Age Normal
Valuation rate of interest	$i=0.01$
Actuarial liability	$AL=451\%$ of payroll
Normal cost	$NC=14.5\%$ of payroll
Earned rates of return	$\{r_t,\ t\geq 1\}$ i.i.d. with $\mathrm{E}r_t=r=0.01$

Table 1
Coefficients of variation of $F(\infty)$ and $C(\infty)$ ($\mathrm{E}r(t) = 0.01$, $\sigma = [\mathrm{Var}\ r(t)]^{1/2}$).

m	$\sigma = 0.025$		$\sigma = 0.05$		$\sigma = 0.10$	
	$\frac{[\mathrm{Var}\ F(\infty)]^{1/2}}{AL}$	$\frac{[\mathrm{Var}\ C(\infty)]^{1/2}}{NC}$	$\frac{[\mathrm{Var}\ F(\infty)]^{1/2}}{AL}$	$\frac{[\mathrm{Var}\ C(\infty)]^{1/2}}{NC}$	$\frac{[\mathrm{Var}\ F(\infty)]^{1/2}}{AL}$	$\frac{[\mathrm{Var}\ C(\infty)]^{1/2}}{NC}$
1	2.5%	77.0%	5.0%	154.0%	9.9%	307.8%
5	3.7	35.1	7.4	70.3	14.8	141.3
10	4.9	25.5	9.9	51.1	19.9	103.2
20	6.8	18.9	13.7	38.1	28.0	78.1
40	9.7	14.7	19.6	29.9	41.6	63.3

Because $\mathrm{E}r_t = i$, $\mathrm{E}F_t = AL$ and $\mathrm{E}C_t = NC$ for $t \geqslant m$, for any initial conditions. Table 1 contains the limiting coefficients of variation of F_t and C_t, that is to say

$$\lim_{t\to\infty} [\mathrm{Var}\ F_t]^{1/2}/AL$$

and

$$\lim_{t\to\infty} [\mathrm{Var}\ C_t]^{1/2}/NC,$$

for various values of m and $\sigma = [\mathrm{Var}\ r_t]^{1/2}$.

Comments. 1. It is seen that for $\sigma \leqslant 10\%$, the standard deviations of F_∞ and C_∞ are nearly linear in σ. This linearity gradually disappears, though, as σ or m become larger.
2. Within the range of σ and m chosen, no single value of m is 'better' than the others. As m is varied, there is a trade-off between Var F and Var C, e.g. increasing m reduces Var C, but increases Var F.
3. This trade-off is a direct outcome of Proposition 4. However, the following approximate formulas give a more intuitive understanding of the way Var F and Var C vary with m. They are valid when $i = 0$ and $\sigma^2 m$ is small (see proof below):

$$\mathrm{Var}\, F_\infty \approx \sigma^2 \frac{m}{3} AL^2, \tag{14}$$

$$\mathrm{Var}\ C_\infty \approx \sigma^2 \frac{1}{m} AL^2. \tag{15}$$

In words: when i is close to 0, the standard deviation of F (resp. of C) is roughly proportional to $m^{1/2}$ (resp. to $1/m^{1/2}$). For instance, in Table 1, moving from $m = 5$ to $m = 20$ approximately doubles st. dev. F_∞ and halves st. dev. C_∞.

Proof of equations (14) and (15). Set $i = 0$ in Proposition 4 to get

$$V_\infty = \sigma^2 AL^2 \Big/ \left(1 - \sigma^2 \sum_{k=1}^{m-1} [(m-k)/m]^2\right),$$

$$\mathrm{Var}\ F_\infty = V_\infty \sum_{k=0}^{m-1} [(m-k)/m]^2,$$

$$\mathrm{Var}\ C_\infty = V_\infty / m.$$

First,

$$\sum_{k=1}^{m-1} [(m-k)/m]^2 = \sum_{j=1}^{m-1} j^2/m^2$$
$$= [(m-1)m(2m-1)/6]/m^2$$
$$\approx m/3.$$

This shows that $V_\infty \approx \sigma^2\ AL^2$ if $\sigma^2 m$ is small. Observing that similarly

$$\sum_{k=0}^{m-1} [(m-k)/m]^2 \approx m/3,$$

we obtain equations (14) and (15). □

Remark. As approximations for Var F_∞ and Var C_∞, equations (14) and (15) are sometimes valuable, even when $i \neq 0$. For example, if $i = 0.01$, $\sigma = 0.05$ and $m = 10$, equation (14) yields

$$[\mathrm{Var}\ F_\infty]^{1/2}/AL \approx 9.1\%,$$

while the exact number if 9.9% (Table 1).

Acknowledgment

This paper is based on part of my Ph.D thesis. I wish to thank my supervisor, Prof. Steve Haberman, of the City University.

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