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On discounting when rates of return are random

Daniel Dufresne, Canada

Routelabille. - Mais aujourd'hui il s'agit de faire mieux qu'une oeuvre de policier... *mieux que ce qu'enseigne l'expérience!... Il s'agit d'être logique*, mais logique, entendez-moi bien, comme le bon Dieu a été logique quand il a dit: 2 + 2 = 4!... IL S'AGIT DE PRENDRE LA RAISON PAR LE BON BOUT!

Gaston Leroux, *Le mystère de la chambre jaune*, Chapitre 7.

1 INTRODUCTION

Consider the following situation: payments $P_0,P_1,...,P_n$ are made at times $0,1,...,n$, and the rates of return used to discount these payments are $R_1,R_2,...,R_n$. It is supposed that the payments and rates of return are random variables. As a consequence, the discounted value of $\{P_k\}$, denoted by X, is also a random variable. This paper deals with the determination, in general approximate, of the distribution of X.

A concrete example of this situation is the terminal funding of pension liabilities: an insurance company sells a contract providing the payment of pensions to a closed group of people. The company invests the proceeds from the sale in bonds, stocks, etc. The returns on these investments are uncertain, and so are the payments to be made. It is immediately clear that in this case the payments are indeed random, and cannot be assumed to be independent from one year to the next. It may be correct to suppose independence between individuals, but the fact that someone is alive or not at one time certainly influences whether or not a payment will be made at all subsequent times. The rates of return on the investments are also uncertain. The distribution of the rate of return in one particular year is influenced by the investments and strategy chosen. These factors also influence the dependence of the rates from one period to the next. For instance, a strategy of buying high quality long term bonds and holding them to maturity would imply of high degree of dependence between yearly rates of return. On the contrary, investing in a small number of volatile stocks might produce less dependence over time. It will be seen that the dependence or independence of the $\{R_k\}$ has a significant impact on the distribution of X. In other words, to let $\{R_n\}$ be independent "as a first approximation" may very well lead to significant errors. As to the relationship between $\{P_k\}$ and $\{R_k\}$, it will be supposed that the two sequences are independent one from the other (except in Section 6, where products such as universal life are considered).

There are a small number of cases, each involving independent payments stretching over an infinite time period, where the exact distribution of X can be calculated (see Dufresne, 1991a, and Section 6 below). In general, however, it is possible only to approximate the distribution of X, either via simulations, or else by fitting a known density function using the first few moments. This is why almost all the papers on randomly discounted payments have dealt only with the moments of X. Three classes of stochastic processes have been used to model the rates of return $\{R_k\}$:

(a) Autoregressive-moving average (ARMA) processes (e.g. Pollard, 1971, Panjer and Bellhouse, 1980, Dhaene, 1990).

(b) Indepedendent and identically distributed (i.i.d.) random variables (e.g. Wilkie, 1976, Boyle, 1976, Waters, 1978).

(c) Moving average (MA) processes (e.g. Dufresne, 1990, Frees, 1991).

Early efforts concentrated mainly on ARMA processes, which provide the greatest flexibility when fitting a process to empirical data. The ARMA case is nevertheless the most difficult to treat. This early emphasis on a difficult problem probably explains why a comparatively simpler one, that of i.i.d. rates of return, took so long to solve (see Section 2). It is only recently that MA processes have appeared in the literature. These processes are simpler to deal with than ARMA processes, while retaining some dependence between time periods.

Concerning payment patterns, almost all previous studies have dealt with annuities-certain and life annuities. Whenever considering a portfolio of annuities, the approach was nearly always to get the moments of the overall discounted value from those of the individual annuities (a notable exeption is Papatriandafylou and Waters, 1984).

This paper describes methods to calculate the moments of a discounted value when rates of return form an i.i.d. sequence, an MA(1) or an MA(2) process. Two specific points are set forth. First, the technique of time reversal is shown to greatly simplify the mathematics, by transforming discount into accumulation. Second, it is suggested that the two processes $\{P_k\}$ and $\{R_k\}$ be separated as much as possible. The moments of the discounted value of a portfolio of annuities are determined only after the moments of the (undiscounted) payments have been found.

Section 2 deals with the i.i.d. case and explains the time reversal argument in its simplest form. Section 3 then applies the same argument to the MA(1) and MA(2) cases. Section 4 briefly describes how all the moments of prospective losses (reserves) and life annuities can be calculated. Section 5 looks at the effect of the dependence structure of the rates of return on the distribution of discounted values. Finally, Section 6 examines whether the normal approximation is appropriate for discounted sums, when discount factors are random.

2 INDEPENDENT AND IDENTICALLY DISTRIBUTED RATES OF RETURN

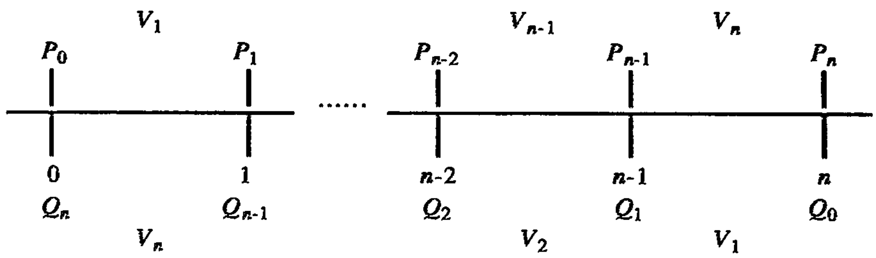
The discounted value of payments $P_0, P_1, \ldots, P_n$, if rates of return are R_1 in the first period, R_2 in the second,...,R_n in the n-th period, is

$$X = P_0 + \sum_{k=1}^{n} V_1 \ldots V_k P_k \,, \tag{1}$$

where $V_k = 1/(1+R_k)$. It is not impossible to calculate the moments of X by simply raising the above equation to the required power and then taking expectations. This section describes an alternative approach, which has a number of advantages over the direct approach.

Instead of discounting each P_k separately back to time 0, work backwards, year by year, from time n to time 0. The initial amount, at time n, is P_n. Moving back one period, the discounted value of P_n, at time $n-1$, plus the payment to be made at that moment is $P_n V_n + P_{n-1}$. Going back one more period, the discounted value of P_{n-2}, P_{n-1} and P_n at time $n-2$ is V_{n-1} times $P_n V_n + P_{n-1}$ plus P_{n-2}, that is to say $P_n V_n V_{n-1} + P_{n-1} V_{n-1} + P_{n-2}$. Moving back $n-2$ more periods yields expression (1)

FIGURE 1



This procedure is formalized in the following way. Relabel the payments $\{Q_n\}$, starting with the last one, i.e. $Q_0 = P_n$, $Q_1 = P_{n-1}, \ldots, Q_n = P_0$. The discount factors $\{V_k\}$ are also taken in reverse, and new symbols could be used. However, the fact that they are independent and identically distributed implies that the order in which they are taken is unimportant, and, in particular, $(V_1, \ldots, V_n)$ has the same distribution as $(V_n, \ldots, V_1)$. Therefore a new process, representing the procedure described in the last paragraph, is defined recursively as

$$B_0 = Q_0$$

$$B_k = V_k B_{k-1} + Q_k\,, \quad k = 1,\ldots,n\,. \tag{2}$$

Then B_n has the same distribution as X. For instance, if $n=2$ then

$$\begin{aligned} X &= P_0 + V_1P_1 + V_1V_2P_2\,, \\ B_2 &= V_2V_1Q_0 + V_2Q_1 + Q_2 \\ &= V_2V_1P_2 + V_2P_1 + P_0 \end{aligned}$$

have the same distribution, since the only difference between them is the order in which the V's appear.

The advantage of the backwards equation (2) over expression (1) is that recursive equations can now be used to calculate the moments. Recalling that $\{P_k\}$ is independent of $\{V_k\}$, we first obtain

$$EB_k = v_1 EB_{k-1} + EQ_k\,,$$

where $v_1 = EV_k$. Next, squaring both sides of (2) and taking expectations,

$$EB_k^2 = v_2 EB_{k-1}^2 + 2v_1 EB_{k-1}Q_k + EQ_k^2\,.$$

The only unknown quantity on the right hand side of this equation is $EB_{k-1}Q_k$. This can be found recursively from

$$EB_0Q_k = EQ_0Q_k$$

$$EB_jQ_k = v_1 EB_{j-1}Q_k + EQ_jQ_k\,, \quad 1 \le j \le k-1\,.$$

Therefore, the computation of the second moment of X requires knowledge of $EQ_jQ_k = EP_{n-j}P_{n-k}$ for $1 \le j \le k \le n$. Similarly, the third order moments $EQ_iQ_jQ_k$ are required to compute the third moment of X, and so forth for higher moments.

A number of observations can be made:

(a) Time reversal transforms discounting into accumulating: the accumulated value of payments $\{P_k\}$ satisfies the recursive equation

$$S_k = (1+R_k)S_{k-1} + P_k$$

which is of the same form as Eq. (2).

(b) The procedure described involves computing the moments of the discount factors and of the payments separately, and then bringing them together to find the moments of the discounted value. This may be an advantage when using multiple scenarios.

(c) Explicit formulas for the moments of annuities are not needed. Programming recursive equations is often less time consuming than using long formulas.

(d) Eq. (2) can also be used to find the distribution of X either via simulations or by numerically performing the required convolutions (first product or V_k and B_{k-1}, then sum of V_kB_{k-1} and Q_k). The latter approach is more complex when there is dependence between the payments.

3 MA RATES OF RETURN

Suppose now that $\{Z_k\}$ is an i.i.d. sequence, and that

$$-\log V_k = m + Z_k + aZ_{k-1}\,, \quad a \neq 0\,.$$

The geometric rates of return form an MA(1) process (the MA(2) case will be dealt with separately, see below). In many statistical applications it is required that the Z's be zero-mean normal variables, and that $|a| < 1$; these conditions are *not* imposed here. The only requirement is that the moment generating function of Z ,

$$w_y = Ee^{-yZ_k}\,,$$

exists whenever it appears in the formulas.

Time reversal is applied in the following fashion: since $(Z_0,\dots,Z_n)$ has the same distribution as $(Z_n,\dots,Z_0)$,

$$X = P_0 + P_1 \exp - (m+Z_1+aZ_0) + \dots + P_n \exp - [nm + \sum_{j=1}^{n} (Z_j+aZ_{j-1})]$$

has the same distribution as

$$B_n = Q_n + Q_{n-1} \exp - (m+Z_{n-1}+aZ_n) + \dots + Q_0 \exp - [nm + \sum_{k=1}^{n} (Z_{k-1}+aZ_k)]\,.$$

As in the previous section, the variables $\{Q_k\}$ are the payments $\{P_k\}$ taken in reverse order. B_n can be obtain recursively from

$$B_k = \exp - (m+aZ_k+Z_{k-1})B_{k-1} + Q_k\,, \quad 1 \le k \le n\,,\, B_0 = Q_0\,.$$

As it stands, this equation does not permit the calculation of moments (not even the first), since Z_{k-1} and B_{k-1} are dependent. There is a simple way out: define

$$C_j = e^{-Z_j}B_j\,, \quad 0 \le j \le n - 1\,.$$

Then

$$B_k = \exp - (m+aZ_k)C_{k-1} + Q_k$$

$$C_k = \exp - [m+(1+a)Z_k]C_{k-1} + e^{-Z_k}Q_k\,\,.$$

The second equation is obtained by multiplying the first one by $\exp - Z_k$. It is seen that C_{k-1} depends on $Q_0,\ldots,Q_{k-1}$ and $Z_0,\ldots,Z_{k-1}$ only; C_{k-1} is thus independent of Z_k, which means that the second equation allows the calculation of the moments of $\{C_k\}$. The moments of $\{B_k\}$ can then be obtained from the first equation. For mean values we can thus use the recursive equations

$$EB_k = e^{-m}w_a EC_{k-1} + EQ_k$$

$$EC_k = e^{-m}w_{1+a}EC_{k-1} + w_1 EQ_k\,.$$

Second moments are obtained from

$$EB_k^2 = e^{-2m}w_{2a}EC_{k-1}^2 + 2e^{-m}w_a EC_{k-1}Q_k + EQ_k^2$$

$$EC_k^2 = e^{-2m}w_{2+2a}EC_{k-1}^2 + 2e^{-m}w_{2+a}EC_{k-1}Q_k + w_2 EQ_k^2\,.$$

The moments $EC_{k-1}Q_k$ can be found recursively by taking expectations on both sides of

$$C_jQ_k = \exp - [m+(1+a)Z_j]C_{j-1}Q_k + e^{-Z_j}Q_jQ_k\,, \quad 1 \le j \le k - 1\,.$$

Let us turn now to rates of return which form an MA(2) process:

$$-\log V_k = m + Z_k + aZ_{k-1} + bZ_{k-2}\,,\, b \ne 0\,.$$

Using the same reasoning as before, $(Z_{-1},\ldots,Z_n)$ has the same distribution as $(Z_n,\ldots,Z_{-1})$ and therefore X has the same distribution as

$$B_n = Q_n + Q_{n-1} \exp - (m+Z_{n-2}+aZ_{n-1}+bZ_n) +$$

$$+ \ldots + Q_0 \exp - [nm + \sum_{j=1}^{n} (Z_{j-2}+aZ_{j-1}+bZ_j)] .$$

The recursive calculation of moments from

$$B_k = \exp - (m+bZ_k+aZ_{k-1}+Z_{k-2})B_{k-1} + Q_k , \quad B_0 = Q_0$$

will require more auxiliary variables than in the MA(1) case. Define C_{-1} , D_{-1} as zero and

$$C_k = \exp - (aZ_k+Z_{k-1})B_k$$

$$D_k = \exp(-Z_k)C_k$$

$$F_{k-1} = \exp(-Z_{k-1})Q_k$$

for $k \geq 0$. Then

$$B_k = \exp - (m+bZ_k)C_{k-1} + Q_k$$

$$C_k = \exp - [m+(a+b)Z_k]D_{k-1} + \exp(-aZ_k)F_{k-1}$$

$$D_k = \exp - [m+(1+a+b)Z_k]D_{k-1} + \exp - [(a+1)Z_k]F_{k-1}$$

$(k\geq 0)$ are suitable for the calculation of the moments of B_n .

4 RESERVES

In this section explicit formulas are found for the moments of prospective losses (reserves) when premiums and death benefits are constant. The same results apply to annuities.

Consider a life age x at issue of the contract. After t years, if the insured is still alive, the prospective loss on the contract is

$${}_tL = cV_1 \ldots V_{J+1} - \pi Y_{J+1} .$$

Here the death benefit is c units, the annual premium is π, J is the curtate future lifetime, and Y_{J+1} stands for the discounted value of one unit payable at the beginning of each year for the rest of the insured's life. When V_t is equal to v, the above definition for ${}_tL$ reduces to the classical (constant interest) formula, see for example Bowers et al, Chapter 7.

To calculate the moments of ${}_tL$, fix $J = j$ and define $P_k = -\pi$, $0 \leq k \leq j$, $P_{j+1} = c$. We then have ${}_tL = X_{j+1}$. The previous sections show how to calculate all the moments of X_{j+1} (recall that j is fixed). All the moments of ${}_tL$ can thus be found by summing with respect to j:

$$E({}_tL)^r = \sum_{j=0}^{\infty} E(X_{j+1}^r)P\,(J=j)\,.$$

In the i.i.d. case we find $B_0 = c$ and

$$B_k = V_k B_{k-1} - \pi\,, \qquad 1 \leq k \leq j+1\,.$$

The process B works backwards from the end of the year of death. First moments satisfy

$$EB_k = v_1 EB_{k-1} - \pi\,, \qquad v_1 = EV_k\,,$$

and thus

$$E({}_t L|J=j) = cv_1^{j+1} - \pi\ddot{a}_{\overline{j+1}|}\,.$$

Summing over j yields

$$E({}_tL) = cA_{x+k} - \pi\ddot{a}_{x+k}\,,$$

where A_{x+k} and $\ddot{a}_{x+k}$ are evaluated at rate $i_1 = 1/v_1 - 1$. This corresponds to the case of a constant rate of return i_1. Second moments (for fixed j) satisfy

$$EB_k^2 = v_2 EB_{k-1}^2 - 2\pi v_1 EB_{k-1} + \pi^2\,.$$

It can be shown (Dufresne, 1991a, 1991b) that this leads to

$$EB_{j+1}^2 = d_{20} + d_{21}v_1^{j+1} + d_{22}v_2^{j+1}$$

where the constants d_{20}, d_{21} and d_{22} depend on c, π and on the v's. Summing over j yields

$$E({}_tL)^2 = d_{20} + d_{21}\,{}^1A_{x+k} + d_{22}\,{}^2A_{x+k}\,.$$

In general,

$$E({}_tL^r) = d_{r0} + \sum_{l=1}^{r} d_{rl}\,{}^lA_{x+k} \tag{3}$$

where $\{d_{rl}\}$ are constants and ${}^lA_{x+k}$ is evaluated at rate $i_l = 1/v_l - 1$.

The above considerations apply almost word for word to the MA(1) case. In the MA(2) case, however, a slight modification has to be made. It is still true that

$$EB_{j+1}^r = d_{r0} + \sum_{l=1}^{r} d_{rl} v_l^{j+1}$$

(here $v_l = E\exp - [l(1+a+b)z_k] = w_{1+a+b})$, but only for $j \geq 1$. For example, if $\pi = 0$ (paid-up life insurance)

$$EB_1 = w_b w_a w_1$$

$$EB_{j+1} = w_b w_{a+b} w_{1+a} w_1 w_{1+a+b}^{j-1}\,, \quad j \geq 1$$

and thus EB_{J+1} is not exactly equal to a constant time A_{x+k} evaluated at rate $i_1 = 1/w_{1+a+b} - 1$. In general this implies that in the MA(2) case formula (3) still holds, but with the difference that d_{r0} now depends on the distribution of J, the curtate future lifetime.

Annuities do not have to be dealt with separately, simply set $\pi = -1$ and $c = 0$ in the above analysis. Finally, we have the following result: *for annuities and prospective losses, when premiums (payments) and death benefits are constants, the rth moment is given by expression (3). The* $\{d_{rl}, 0 \leq l \leq r\}$ *are constants independent of the law of mortality when the rates or return are i.i.d. or* MA(1). *When they are* MA(2), *however,* d_{r0} *does depend on the distribution of future lifetime at age* $x + k$.

It is the author's opinion that explicit formulas for the moments of reserves or annuities are of little use (except for mean values) when rates or return are random. This is because the reserves on different lives are now dependent random variables, so that, for example, variances cannot be added together directly. Waters (1978), Papatriandafylou and Waters (1984) and Frees (1991) have devised techniques to find the variance of a sum of contracts. These techniques deal with random mortality and random rates of discount simultaneously. Nevertheless, it is probably best in most cases to separate the two problems (mortality and discount rates) in the way suggested in Sections 3 and 4. First determine the moments of the undiscounted payments $\{P_k\}$; this involves mortality only. Then apply time reversal to introduce random discounting.

5 TIME DEPENDENCE OF THE DISCOUNT FACTORS

Suppose G is $N(m,s^2)$. Then

$$Ee^{-G} = \exp(-m+s^2/2) .$$

If G is the (geometric) rate of return during a certain period, then the mean discount factor for the period, Ee^{-G} , is not equal to the exponential of the mean of G (i.e. e^{-m}) . In fact, the constant geometric rate which will reproduce average discounted values is $m - s^2/2$, which is smaller than $m = EG$. This relationship will hold given any distribution for G : since the function e^{-x} is convex,

$$Ee^{-G} \geq e^{-EG}$$

(from Jensen's inequality) and so

$$-\log E\, e^{-G} \leq EG . \tag{4}$$

The inequality is strict as soon as G is not degenerate (i.e. equal to a constant). Using the cumulant generating function of G (Cramer, 1945, p. 185) the difference between the two sides of (4) can be expressed as an infinite series, the first term of which is half the variance of G :

$$-\log Ee^{-G} = EG - \tfrac{1}{2}\text{Var}\, G + \text{(series involving higher order cumulants)}.$$

The above considerations apply to one period, say one year. Next, let us turn to n periods. If the geometric rates of return are $G_1,\ldots,G_n$, then the overall rate for the n periods is $G_1 + \ldots + G_n$. The constant annual rate of return which will reproduce the mean value of a unit discounted for n periods is therefore

$$-\frac{1}{n}\log E\,\exp - \sum_{k=1}^{n} G_k = \frac{1}{n}\sum_{k=1}^{n} EG_k - \frac{1}{2n}\text{Var}\sum_{k=1}^{n} G_k + (\ldots) . \tag{5}$$

The variance of the sum of the $\{G_k\}$ involves their covariances. Therefore it can be expected that the dependence between the $\{G_k\}$ will affect the distribution of discounted values, even when EG_k and Var G_k are unchanged.

Table 1 illustrates this point, when rates of return are MA(1). The table shows the expected discounted values of one unit twenty years hence (Y_{20}) and of a twenty-year annuity-certain, with payments at the end of the year (X_{20}). If G_k is equal to $Z_k + aZ_{k-1}$, then the discounted value of one unit in n years' time is

$$Y_n = \exp - (n \cdot m + Z_1 + aZ_o + \ldots + Z_n + aZ_{n-1})$$

and has expected value

$$EY_n = e^{-n \cdot m} w_a w_1 w_{1+a}^{n-1}, \quad n = 1,2,\ldots \tag{6}$$

Thus

$$EX_n = e^{-m} w_a w_1 (1 - w_{1+a}^n e^{-n \cdot m})/(1 - w_{1+a} e^{-m}) . \tag{7}$$

The variables $\{Z_k\}$ are assumed normal with mean zero and variance s^2. The average rate or return (m) is equal to 10%. Observe that the variance of one rate of return, and the correlation coefficient between two consecutive ones, are respectively

$$\text{Var}(Z_k + aZ_{k-1}) = (1+a^2)s^2$$

$$r = \text{Cov}(Z_{k+1} + aZ_k, Z_k + aZ_{k-1})/(1+a^2)s^2 = a/(1+a^2) .$$

In Table 1, the variance of $G_k = m + Z_k + aZ_{k-1}$ is kept equal to .01 . However, the parameters a and s^2 are varied so as to obtain different correlations between consecutive rates. The mean value m is set equal to .10 .

When $a = 0$ the rates of return are i.i.d. As pointed out at the beginning of this section, average discounted values correspond to a constant rate equal to

$$j = m - s^2/2 = .10 - .01/2 = .0950 .$$

The situation is different when the rates are dependent. The equivalent constant rate j turns out to be greater when $a < 0$ (negative correlation) and smaller when $a > 0$ (positive correlation). Indeed, Eqs. (6) and (7) show that

$$j \approx m - \log w_{1+a} = m - (\tfrac{1}{2} + r)s^2 \tag{8}$$

TABLE 1. MEAN DISCOUNTED VALUES, WHEN CORRELATION BETWEEN RATES OF RETURN VARIES

Assumptions

Rates of return	MA(1) process: $G_k = .10 + Z_k + aZ_{k-1}$
Correlation	r = correlation of G_k and G_{k-1}
Mean rate of return	.10 in all cases
Variance of rates of return	.01 in all cases
Y_{20}	Discounted value of one unit 20 years from now
X_{20}	Discounted value of a 20 year annuity-certain, with payments at the end of the year
Equivalent constant rate (j)	Rate which reproduces mean value of annuity (X_{20})

a	r	EY_{20}	EX_{20}	j
1.00	.500	.1645	8.819	.0907
.75	.480	.1639	8.807	.0908
.50	.400	.1614	8.761	.0915
.25	.235	.1564	8.666	.0930
0	0	.1496	8.533	.0950
-.25	-.235	.1430	8.404	.0970
-.50	-.400	.1386	8.316	.0985
-.75	-.480	.1365	8.273	.0992
-1.00	-.500	.1360	8.263	.0993

Table 1 clearly shows (1) that the fact that rates of return are random has to be taken into account, even when calculating average discounted values, and (2) that the dependence between the rates has a further effect on the distribution of discounted values, in particular on the average discounted value.

Remark. Approximation (8) is based on MA(1) rates of return; it does not hold for other processes. When rates are dependent, the equivalent rate j varies slightly with the pattern of discounted payments. See also Frees (1991) on this topic. □

For a similar analysis of accumulated values, when arithmetic rates of return are MA(1), see Dufresne (1990).

6 THE NORMAL APPROXIMATION

This section concerns the normal approximation, and its appropriateness when rates of return are random. It turns out that the normal approximation may be worse when rates are random than when they are constant.

Consider the prospective loss (reserve) for a portfolio of insurance policies. Suppose that, if future rates of return on investments were known, the benefits payable under different policiers would be independent. Examples of this situation are

(a) property and casualty insurance, when conditions for independence (e.g. different geographical locations) are satisfied;

(b) terminal funding of pension liabilities;

(c) variable (or universal) life insurance.

The last example is of a slightly different nature, since the benefit payouts often explicitly depend on the future rates of return that the insurance company can achieve, and possibly also on other (uncertain) financial information (e.g. what other companies will be offering). Nevertheless, once this financial information is known, in most cases the remaining "randomness", for instance time of death, can be assumed to affect different policies independently.

Suppose future rates of return (as well as other relevant financial information) are known with certainty. Denote the prospective losses on the portfolio's policies by $X_1,...,X_n$, and the total prospective loss by $S_n = X_1 + ... + X_n$. Then the distribution of $(S_n\text{-}ES_n)/(\text{Var } S_n)^{1/2}$ is approximately standard normal, i.e.

$$\text{distribution of } S_n \approx \text{N}(m,s^2)$$

if n is "large". How large n has to be depends on the individual distributions of the $\{X_i\}$, and on the degree of accuracy required. The theoretical justification is Lindeberg's Central Limit Theorem (e.g. Feller, 1971, p. 518). Lindeberg's condition means, roughly speaking, that the contribution of each of the $\{X_i\}$ should be "negligible". Elementary illustrations can be found in Section 11.5 of Bowers et al (1986).

Denote by Q the vector of rates of return and other financial information affecting future payments. Then the preceding analysis is valid for any fixed $Q = q$. For given q we get

$$S_n \approx \text{N}(m(q),s^2(q)) .$$

The distribution function of S_n can therefore be approximated by weighting these normal distributions by the distribution of Q :

$$P(S_n \le x) \approx \int \Phi([x-m(q)]/s(q))dH(q)$$

where Φ is the standard normal distribution function and H is the distribution function of Q. This is justified, at least when $m(q) = 0$, by the following result.

Proposition 1. Suppose that for each fixed $Q = q$, the variables $\{X_i, i \ge 1\}$ are i.i.d. with mean zero and variance $\sigma^2(q)$. Let $T_n = X_1 + ... + X_n$. If H is the distribution function of Q, then

$$\lim_{n\to\infty} P(T_n/\sqrt{n} \le x) = \int \Phi(x/\sigma(q))dH(q)$$

(This is Exercise 21, p. 287, of Feller (1971).)

The approximate distribution given for S_n can be obtained either by direct integration or by simulation. The first approach can be quite difficult, as the vector Q can easily be of dimension 20 or 30. The other one, used by Frees (1991), is usually preferable.

Another possibility (e.g. Atkinson, 1990, p. 92) is to calculate the overall (i.e. unconditional) mean and variance of S_n, and then apply the normal approximation. Somehow this is an appealing idea, as the distribution of S_n is a weighted average of normal distributions.

The problem with this approach is that a weighted average of normal distributions is in general not normal. In other words, for each q the normal approximation may have a certain accuracy, but the overall approximation, after summing with respect to $H(q)$, could actually be worse. A theoretical justification of this possibility follows.

Proposition 2. The weighted average of normal distributions

$$\int \Phi(x/\sigma(q))dH(q) \tag{9}$$

cannot itself be normal, unless $\sigma^2(Q)$ is a constant.

Proof. Distribution (9) has characteristic function

$$\int \exp - [\tfrac{1}{2} t^2\sigma^2(q)]dH(q) = \mathrm{E} \exp - [\tfrac{1}{2} t^2\sigma^2(Q)]$$
$$\ge \exp - [\tfrac{1}{2} t^2 E\sigma^2(Q)] . \tag{10}$$

The inequality is a consequence of Jensen's inequality (Bowers et al, 1986, p. 9). But the variance of (9) is $E\sigma^2(Q)$, so that, if that distribution is normal, its characteristic function has to be equal to (10). There is thus equality in (10), meaning that $\sigma^2(Q)$ is degenerate (i.e. constant with probability 1). □

As a simple illustration, consider the discounted value of a single amount P having a normal distribution,

$$X = VP$$

where V is independent of P and has distribution $P(V = v) = 1 - P(V = v')$. Then X is normal for each given value of V, but the unconditional distribution of X is certainly not normal.

We conclude that approximating the distribution of discounted values requires more care when rates of return are random, than when they are not. For instance, it may be useful to calculate third or even fourth moments. The paper by Waters (1978) contains illustrations of skewness and kurtosis of distributions of discounted values for a portfolio of life contracts, when rates are i.i.d.

However we may still expect the normal approximation to be reasonably accurate when rates of return do not fluctuate much. The two following examples illustrate this point.

Example A. Consider independent payments $P_1, P_2, \ldots$ having an exponential distribution with parameter (mean) m. Supose that the discount factors $V_1, V_2, \ldots$ are also independent and have density

$$f_V(x) = ax^{a-1}, \quad 0 < x < 1, \, a > 0.$$

The payments and discount factors are assumed mutually independent. The geometric rates of return corresponding to these discount factors can be shown to possess an exponential distribution with mean $1/a$. Then (see Dufresne, 1991a, Section 3), the discounted value of all future payments

$$X = \sum_{k=1}^{\infty} V_1 \ldots V_k P_k$$

has a gamma distribution with parameters a and m.

The gamma distribution approaches the normal as the first parameter increases. We thus see that the distribution of X converges to the normal as a tends to infinity. This corresponds to the geometric rates of return converging to zero:

$$G_k = -\log V_k \sim \exp\,(1/a) \rightarrow 0 \text{ as } a \rightarrow \infty.$$

Vervaat (1979) generalizes this fact to other distributions for the discount factors and payments. If we replace the geometric rates of return $\{G_k\}$ with $\{G_k/c\}$, we always get a normal distribution as $c \to \infty$. □

Example B. In this example the rates of return are random, but the payments are constant. Nevertheless, it is still true that the distribution of the discounted value approaches the normal as the variance of the rates of return decreases to zero.

Consider the random counterpart of

$$\bar{a}_{\overline{\infty}|} = \int_0^\infty e^{-\delta t} dt .$$

Let the sum of the (geometric) rates of return over the period from 0 to t form a brownian motion W_t will mean δt and variance $\sigma^2 t$. This is the continuous counterpart of i.i.d. rates of return in discrete time. The discounted value of one unit per annum payable continuously over an unlimited period becomes

$$Y = \int_0^\infty \exp - (W_t) dt .$$

If $\delta > 0$, this integral converges and has distribution

$$\frac{1}{Y} \sim \Gamma(2\delta/\sigma^2, \sigma^2/2) .$$

(see Dufresne, 1991a, Section 4).

Proposition 3. The distribution of

$$U = (Y - EY)/(\mathrm{Var}\ Y)^{1/2}$$

converges to $N(0,1)$ as $b = 2\delta/\sigma^2 \to \infty$.

Proof. Find the density of U, and then use Stirling's formula to show that it converges to that of the standard normal. □

This indicates that the normal approximation improves as the ratio δ/σ^2 increases, i.e. as the variance of the rates of return decreases relative to the mean. This is rather unexpected: *a priori* one would think that the absolute size of the variance would be the most important factor, but this is not so. (In fact, the same thing happened in Example A, since (1) the ratio mean/variance of an exponential distribution with parameter $1/a$ is equal to a, and (2) the normal distribution is obtained as $a \to \infty$.) □

We tentatively conclude that the normal approximation may not be appropriate when the variance of the rates of return is large relative to the mean. It is of course impossible to lay down precise rules as to how large the ratio mean/variance should be, but an examination of the skewness coefficients in Examples A and B may be helpful.

In Example A, the ratio mean over variance is equal to a, while the skewness coefficient of X is $2/\sqrt{a}$. For example, if $EG_k = .05$, i.e. $a = 20$, then the skewness coefficient is $1/\sqrt{5} = .447$. Some values of the skewness coefficient of Y (Example B) are given in Table 2. They are based on the formula

$$\text{skewness } (Y) = 4(b\text{-}2)^{1/2}/(b\text{-}3), \quad b = 2\delta/\sigma^2 > 3 .$$

TABLE 2. SKEWNESS COEFFICIENT (g) OF Y (EXAMPLE B)

Mean (δ)	Standard Deviation (σ)	δ/σ^2	g
.02	.01	200	.201
.02	.10	2	5.657
.05	.01	500	.127
.05	.10	5	1.616
.08	.01	800	.100
.08	.10	8	1.151

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RÉFÉRENCES

Atkinson, D.B. (1990). Introduction to pricing and asset shares. Study Note 210-25-90, Society of Actuaries.

Bowers, N.L. et al (1986). Actuarial Mathematics. Society of Actuaries, Itasca, Illinois.

Boyle, P.P. (1976). Rates or return as random variables. Journal of Risk and Insurance 43: 693-713.

Cramer, H. (1945). Mathematical Methods of Statistics. Princeton University Press.

Dhaene, J. (1989). Stochastic interest rates and autoregressive integrated moving average processes. Astin Bulletin 19: 131-138.

Dufresne, D. (1990). Fluctuations of pension contributions and fund levels. Actuarial Research Clearing House 1990.1: 111-120.

Dufresne, D. (1991a). The distribution of a perpetuity, with applications to risk theory and pension funding. Scandinavian Actuarial Journal, in press.

Dufresne, D. (1991b). Discussion of "Stochastic life contingencies with solvency considerations", by E.W. Frees, Transactions of the Society of Actuaries, in press.

Feller, W. (1971). An Introduction to Probability Theory and its Applications. Wiley, New York.

Frees, E.W. (1991). Stochastic life contingencies with solvency considerations. Transactions of the Society of Actuaries, in press.

Panjer, H.H. and Bellhouse, D.R. (1980). Stochastic modelling of interest rates with applications to life contingencies. Journal of Risk and Insurance 47: 91-110.

Papatriandafylou, A. and Waters, H.R. (1984). Martingales in life insurance. Scandinavian Actuarial Journal 1984: 210-230.

Pollard, J.H. (1971). On fluctuating interest rates. Bulletin de l'Association Royale des Actuaires Belges 66: 68-94.

Vervaat, W. (1979). On a stochastic difference equation and a representation of non-negative infinitely divisible random variables. Adv. Appl. Prob. 11: 750-783.

Waters, H.R. (1978). The moments and distributions of actuarial functions. Journal of the Institute of Actuaries 105: 61-75.

Wilkie, A.D. (1976). The rate of interest as a stochastic process: Theory and applications. Proceedings of the 20th International Congress of Actuaries 1: 325-338.

MAIN CONCLUSIONS

This paper examined the distribution of the discounted value of future random payments, when the discount factors are also random. The paper showed that the moments of such distributions can be calculated recursively, when the rates of return form a moving average process of order less than or equal to two. Using the same techniques, explicit formulas for the moments of annuities or reserves can also be calculated. However, it was noted that these explicit formulas are of little use, since the discounted value of payments under two separate contracts are dependent variables. As a consequence, it is better, when possible, to deal with aggregate payments and discount factors separately. Further analysis has also showed that the dependence between successive rates of return has a significant effect on the distributions of discounted values. Finally, an examination of the normal approximation has indicated that it may be worse when rates of discount are random, than when they are constant.

SUMMARY

Some new results are presented concerning the distributions of discounted values when rates of discount as well as payments are random. The paper first deals with the moments of such discounted values. The technique of time reversal (i.e. moving backwards from the last payment) is shown to greatly simplify the calculations. The effects of the dependence between successive discount rates are examined. Finally, the suitability of the normal approximations is studied. Most of the paper assumed that payments are independent of discount factors, but the last section is applicable to situations where they are dependent, for example universal life.

RÉSUMÉ

De nouveaux résultats sont présentés au sujet des valeurs actuelles lorsque les taux d'intérêt, de même que les versements futurs, sont aléatoires. Premièrement, les moments (moyenne, variance, etc.) de telles distributions sont étudiés. La technique du retournement du temps ("prendre sa raison par le bon bout", comme dirait Rouletabille) simplifie beaucoup les calculs. Ensuite, les implications de la dépendence entre les taux d'intérêt sont analysées. Finalement, l'approximation normale est remise en cause, lorsque les taux d'intérêt sont aléatoires. La plus grande partie de l'article suppose que les versements sont indépendants des taux d'intérêt, mais la dernière section est applicable aux cas où ils ne le sont pas, par exemple lorsqu'on considère des contrats d'assurance-vie universelle.