THE INTEGRATED SQUARE-ROOT PROCESS

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Abstract

This paper studies some of the properties of the square-root process, particularly those of the integral of the process over time. After summarizing the properties of the square-root process, the Laplace transform of the integral of the square-root process is derived. Three methods for the computation of the moments of this integral are given, as well as some properties of the density of the integral. The relationship between the Laplace transforms of a variable and of its reciprocal is studied. An application to the generalized inverse Gaussian distribution is given.

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1. Introduction

The square-root process is the unique strong solution of the stochastic differential equation (SDE in the sequel)

$$dX = (\alpha X + \beta) dt + \gamma \sqrt{X} dW, \quad X_0 = \bar{x} \geq 0,$$

(1.1)

where $\alpha \in \mathbb{R}$, $\beta > 0$, $\gamma > 0$, $\bar{x} \geq 0$, and $W$ is standard Brownian motion. Other names used for the same process are squared Gaussian (since the square of some Gaussian processes can be shown to satisfy (1.1) — a comment attributed to L.C.G. Rogers), Feller (because of the early work of William Feller on the process), CIR (since Cox, Ingersoll & Ross (1985) used (1.1) as a model for the spot interest rate, and gave the formulas for risk-free bonds under this assumption), and Heston model (since Heston (1993) studied the pricing of options when the squared volatility of the underlying security satisfies (1.1)).

The integral of $X$ arises in both volatility and interest rate models. If the stock price follows the process

$$dS = \mu S dt + \sqrt{X} S dV,$$
where \((V,W)\) is two-dimensional Brownian motion (possibly correlated), and \(X\) is the solution of (1.1), then the integral

\[
Y_t = \int_0^t X_s \, ds
\]

is the cumulative variance of the stock returns up to time \(t\). This integrated variance process occurs in volatility swaps (Carr, 2001), and also in other derivatives (see below). The reason for this research is the quest for explicit formulas for options with payoffs involving \(Y_t\). The results in this paper have already been used by the author for this purpose; this will be the subject of a subsequent paper.

The process \(Y_t\) obviously arises in bond pricing, when spot interest rates follow the CIR model. This is because the price of a risk-free zero-coupon bond maturing at time \(t\) equals

\[
E_Q e^{-Y_t}
\]

where \(Q\) is a risk-neutral measure. Another example would be options based on the average spot interest rate, for instance a call on the average of spot rates \(X_t\) over \([0, T]\), with strike \(R_0\):

\[
e^{-\int_0^T X_t \, dt} \left( \frac{1}{T} \int_0^T X_t \, dt - R_0 \right) = e^{-Y_t} \frac{1}{T} (Y_T - TR_0)_+
\]

The price of such an option would be the expectation of this discounted payoff with respect to the risk neutral measure, which requires knowledge of the distribution of \(Y_T\).

The moments and distribution of \(Y_t\) may also be of interest in the statistical estimation of the parameters \(\alpha, \beta, \gamma\) in (1.1).

Here is a summary of the paper. Section 2 shows that the moments of \(X_t\) are solutions of a sequence of ordinary differential equations, and that these moments allow the determination of the moment generating function and of the density of the process. The moments of \(X_t\) are required in the sequel, and the technique used to obtain them is used again in Section 3. Moreover, Section 2 yields an improvement of a result due to Dufresne (1989), regarding the moments of a process related to Asian options and the Courtadon interest rate model. The density of \(X_t\) in (1.1) is of course well known, but the derivation given in Section 2 is not common (the author has not seen it in the literature), and clearly shows that the distribution of the process is the convolution of two distributions, one compound Poisson/Exponential and the other gamma. (The distribution of \(X_t\) has been called “non-central chi-square,” apparently because it reduces to the distribution of the square of non-zero normal variable when \(\alpha = 0\).) The better known way of finding the moment generating function and density of \(X_t\) is via the relationship with Bessel processes, and this is also indicated in Section 2.

Section 3 shows how the moments of the integral of the square-root process may be obtained, either by solving linear differential equations, or more directly by a recursive procedure; the results are apparently new. Section 4 is a derivation of the Laplace transform
of the integral of the square-root process, based on the formula for the price of a zero-coupon bond in the CIR framework (the latter was based on results previously known in probability theory). The Laplace transform yields a third way of calculating the moments. Section 5 extends a result which is used in Section 4, and is of independent interest in Probability Theory. Formulas are given which relate the Laplace transform of a random variable $U > 0$ to the Laplace transform of $1/U$. They extend one well-known formula for Laplace transforms. An application to the generalized inverse Gaussian distribution is given.

2. A sequence of ODEs; the moments and distribution of $X_t$

When the square-root process (1.1) is used as a financial model, it is usually made “mean-reverting” by letting $\alpha < 0$. In the sequel no such restriction is imposed, except that Theorem 2.3 assumes $\alpha \neq 0$.

The moments of $X_t$ are easy to obtain recursively, by applying Ito’s formula and then taking expectations. A general formula for the moments of $X_t$ will now be derived. The technique is similar to the one used in Dufresne (1989, 1990) for the integral of geometric Brownian motion, and yields an extension of one result in Dufresne (1989) (Corollary 2.2 below). Next, the Laplace transform of the density of $X_t$ is derived, based on the moments only. This derivation is different from the usual ones, which are either based on PDEs, or on the relationship with Bessel processes. Our derivation is based on the assumption that all the moments of $X$ are finite. The moments are calculated and then summed to get the moment generating function (MGF). Finally, the density is exhibited, and is shown to be the convolution of a Gamma and a compound Poisson/exponential.

(N.B. The MGF of a probability measure $m$ is the mapping

$$r \mapsto \int e^{rx} m(dx).$$

The concept is thus the same as the Laplace transform, but $a+$ instead of $a−$ in front of the argument.)

Applying Ito’s formula with $f(x) = x^k$ to (1.1), we get

$$dX^k = (a_k X^k + b_k X^{k-1}) dt + \gamma k X^{k-\frac{1}{2}} dW_t$$

with

$$a_k = \alpha k, \quad b_k = \beta k + \frac{1}{2} \gamma^2 k(k-1).$$

Eq. (2.1) is the same as

$$X^k_t = \bar{x}^k + \int_0^t (a_k X^k_s + b_k X^{k-1}_s) ds + \int_0^t \gamma k X^{k-\frac{1}{2}}_s dW_s.$$

Here we let the initial value of the process be $\bar{x} \geq 0$, and we take as given that $E X^n < \infty$ for all $n \geq 0$. The consequence is that the Ito integral above has expected value 0. Defining
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\[ m_k(t) = \mathbb{E} X_t^k, \quad k = 0, 1, \ldots, \] we can then differentiate with respect to \( t \) on each side to obtain

\[ m'_k(t) = a_k m_k(t) + b_k m_{k-1}(t), \quad k \geq 1, \quad t > 0 \]  
\[ m_k(0) = \bar{x}^k, \quad k \geq 0. \]  

Eq. (2.2) is of the same type as Eq. (18) in Dufresne (1989), and can be solved in the same way. The main difference here is that the initial value of \( m_k(t) \) is not 0.

**Theorem 2.1.** Let \( a_0 = 0 \) and suppose the numbers \( \{a_0, \ldots, a_K\} \) are distinct. Then the solution of (2.2), subject to (2.3) and \( m_0(t) \equiv 1 \) is

\[ m_k(t) = \sum_{j=0}^{k} d_{kj} e^{a_j t}, \quad k = 0, \ldots, K, \]  

where

\[ d_{kj} = \sum_{i=0}^{j} \bar{x}^i \left( \prod_{m=i+1}^{k} b_m \right) \frac{1}{\prod_{\ell=i}^{k} a_j - a_\ell}, \quad j = 0, \ldots, k. \]  

**Proof.** Define a product over an empty set of indices as 1. Then formulas (2.4) and (2.5) are correct for \( k = 0 \). For \( k = 1 \), (2.2) implies

\[ m_1(t) = m_1(0) e^{a_1 t} + b_1 e^{a_1 t} \int_0^t e^{-a_1 s} ds = \bar{x} e^{a_1 t} + b_1 \frac{e^{a_1 t} - 1}{a_1} = \frac{-b_1}{a_1} + \left( \frac{b_1}{a_1} + \bar{x} \right) e^{a_1 t}. \]

For \( j = 0 \) and \( j = 1 \), \( m_j(t) \) is a combination of \( e^{a_\ell t} \), \( \ell \leq j \). If this claim is correct for \( j = 0, \ldots, k-1 \), then \( m_k(t) \) is the solution of an inhomogeneous ordinary differential equation with a forcing term equal to a combination of exponentials:

\[ m'_k(t) - a_k m_k(t) = \sum_{j=0}^{k-1} C_j e^{a_j t} \]
\[ \implies m_k(t) = m_k(0) e^{a_k t} + e^{a_k t} \int_0^t e^{-a_k s} \sum_{j=0}^{k-1} C_j e^{a_j s} ds. \]  

(2.6)

Since the \( \{a_0, \ldots, a_K\} \) are distinct, \( m_k(t) \) has to be of the form

\[ d_{k0} + d_{k1} e^{a_1 t} + \ldots + d_{kk} e^{a_k t}. \]

By induction, this is true for \( k = 0, \ldots, K \). As a function of \( \bar{x} \), \( m_k(t) \) is a polynomial of degree \( k \), and the coefficient of \( \bar{x}^k \) in \( m_k(t) \) is \( e^{a_k t} \) (because \( \bar{x}^k \) is the initial condition \( m_k(0) \)).
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Insert (2.4) into (2.2) to obtain

\[ \sum_{j=0}^{k} a_j d_{kj} e^{a_j t} = \sum_{j=0}^{k} a_k d_{kj} e^{a_j t} + \sum_{j=0}^{k-1} b_k d_{k-1,j} e^{a_j t}, \]

which implies

\[ d_{kj} = \frac{b_k}{a_j - a_k} d_{k-1,j} = \frac{b_k}{a_j - a_k} \cdots \frac{b_{j+1}}{a_j - a_{j+1}} d_{jj}, \quad 0 \leq j < k. \quad (2.7) \]

Since \( m_j(t) \) is a polynomial of degree \( j \) in \( \bar{x} \), the last equation implies that \( d_{kj} \) is also a polynomial of degree \( j \) in \( \bar{x} \), which we write as

\[ d_{kj} = \sum_{i=0}^{j} d_{kji} \bar{x}^i. \]

By identifying the coefficients of \( \bar{x}^i \) in (2.7), we get

\[ d_{kji} = \frac{b_k}{a_j - a_k} \cdots \frac{b_{j+1}}{a_j - a_{j+1}} d_{jj}, \quad 0 \leq i \leq j < k. \]

We also know that \( d_{kkk} = 1 \), since the leading coefficient of \( m_k(t) \), as a polynomial in \( \bar{x} \), is \( e^{a_k t} \) (see (2.6)). The missing constants are thus \( d_{kki}, i = 0, \ldots, k-1, k \geq 1 \). All this shows that the problem is the same for each power of \( \bar{x} \): for any \( i = 0, 1, \ldots \), we have

\[ d_{iii} = 1 \quad (2.8a) \]
\[ d_{kji} = \frac{b_k}{a_j - a_k} d_{k-1,j,i}, \quad i \leq j < k \quad (2.8b) \]
\[ \sum_{j=i}^{k} d_{kji} = 0, \quad 0 \leq i < k. \quad (2.8c) \]

(The last equality comes from the initial condition \( m_k(0) = \bar{x}^k \).) Hence, the problem of finding \( \{d_{kj0}; 0 \leq j \leq k\} \) when \( \bar{x} \neq 0 \) is the same as finding \( \{d_{kj}; 0 \leq j \leq k\} \) when \( \bar{x} = 0 \). The latter was done in Dufresne (1989), for the special case \( b_k = k \). The same arguments work, however, for a general sequence \( \{b_k; k \geq 1\} \). The main argument is that

\[ \sum_{j=0}^{k} \prod_{\ell=0}^{k} \frac{1}{a_j - a_\ell} = 0 \quad (2.9) \]

for any distinct numbers \( a_0, \ldots, a_k \). We recall the proof given in Dufresne (1989). Lagrange’s formula for partial fractions decomposition of a rational function is

\[ \frac{P(x)}{Q(x)} = \sum_{j=0}^{k} \frac{1}{x - a_j} \frac{P(a_j)}{Q'(a_j)}, \]
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for polynomials $P$ and $Q$, the latter with $k + 1$ distinct zeros $\{a_0, \ldots, a_k\}$, the former of degree less than or equal to $k$. Let

$$Q(x) = \prod_{j=0}^{k}(x - a_j), \quad P(x) = \frac{y - x}{Q(y)}.$$  

Then

$$P(x) = \sum_{j=0}^{k} \frac{y - a_j}{Q(y)} \frac{Q(x)}{(x - a_j)Q'(a_j)}.$$  

Letting $y \to x$ yields (2.9).

For $i = j = 0$, we find (from (2.8a, b))

$$d_{k00} = \left( \prod_{m=1}^{k} b_m \right) \prod_{\ell=0}^{k} \frac{1}{a_0 - a_\ell}, \quad k \geq 0.$$  

Next, $d_{110} = -d_{100} = b_1/(a_1 - a_0)$ (from (2.8c)), which implies

$$d_{k10} = \frac{b_k \cdots b_2}{(a_1 - a_k) \cdots (a_1 - a_2)} d_{110} = \left( \prod_{m=1}^{k} b_m \right) \prod_{\ell=0}^{k} \frac{1}{a_1 - a_\ell}, \quad k \geq 1.$$  

To prove the result in general, represent the $\{d_{k,j0}\}$ on a grid, with $k$ horizontal and $j$ vertical. For some integer $J > 0$, suppose that

$$d_{k,j0} = \left( \prod_{m=1}^{k} b_m \right) \prod_{\ell=0}^{k} \frac{1}{a_j - a_\ell}, \quad k \geq j, \quad j = 0, \ldots, J - 1,$$  

that is, assume the result holds for the lines $j = 0, \ldots, J - 1$. Then, by (2.8c) and (2.9), (2.10) also holds for $k = j = J$. From this and (2.8b), we find, for $k > J$,

$$d_{k,j0} = \frac{b_k \cdots b_{J+1}}{(a_J - a_k) \cdots (a_J - a_{J+1})} d_{J,j0} = \left( \prod_{m=1}^{k} b_m \right) \prod_{\ell=0}^{k} \frac{1}{a_J - a_\ell}.$$  

By induction, (2.10) applies for all $0 \leq j \leq k$. For the coefficients of $\bar{x}^i, i \geq 1$, the problem is the same, except that it “starts” on line $i$, so that, in the above arguments, $(j, k)$ has to be replaced with $(j + i, k + i)$.

**Remark.** Similar formulas hold when some of the $\{a_0, \ldots, a_K\}$ are identical. The only difference is that terms of the form $t^\ell e^{a_j t}$ make their appearance, where $\ell$ is an integer
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which depends on the number of times the same $a_k$ appears in $\{a_0, \ldots, a_K\}$. For instance, suppose $a_0 = 0, a_1 = a_2 \neq 0$. The solution of (2.2) is then

$$m_2(t) = \bar{x}^2 e^{a_1 t} + b_2 \left( \bar{x} + \frac{b_1}{a_1} \right) t e^{a_1 t} - \frac{b_1 b_2}{a_1^2} (e^{a_1 t} - 1).$$

The $t e^{a_1 t}$ reappears in $m_j(t)$, $j \geq 3$. The solutions of the differential equations nevertheless depend continuously on the parameters $a_0, a_1, \ldots$, so that the solutions when some of them are identical can be found by taking limits in the solutions given in the theorem.

As a first application of Theorem 2.1, we generalize Proposition 2 of Dufresne (1989). Suppose $\{S_t\}$ satisfies

$$dS_t = (\eta S_t + \zeta) dt + \sigma S_t dW_t, \quad S_0 = \bar{x}. \quad (2.11)$$

Then the moments $m_k(t) = E S_t^k$, $k = 0, 1, \ldots$, satisfy Eqs.(2.3-2.4), with

$$a_k = k \eta + k(k - 1) \sigma^2/2, \quad b_k = k \zeta. \quad (2.12)$$

**Corollary 2.2.** If the $\{a_k\}$ are all distinct, the moments of the process in (2.11) are given by (2.4), with $\{a_k\}$ as in (2.12) and

$$d_{kj} = k! \sum_{i=0}^j \bar{x}^i \frac{\zeta^{k-i} i! \prod_{\ell=i}^k 1}{a_j - a_\ell}, \quad j = 0, \ldots, k.$$

**Remark.** The process $S$ in (2.11) has been called the Courtadon interest rate model (Courtadon, 1982). The moments of the spot rate are thus given by Corollary 2.2. The limit distribution of $S_t$ as $t$ tends to infinity (inverse Gamma if $\eta < \sigma^2/2$, $\infty$ otherwise) is derived in Dufresne (1990). In that paper, $S_t$ represents the market value of an initial amount $S_0$ and of a continuous payment stream of $\zeta$ units, all invested in a security represented by a geometric Brownian motion.

**Theorem 2.3.** Suppose $\alpha \neq 0$. If $X_t$ satisfies (1.1), then its moments are

$$E X_t^k = \sum_{j=0}^k \theta_{kj} e^{\alpha j t}, \quad k = 0, 1, \ldots,$$

where

$$\theta_{kj} = \sum_{i=0}^j \bar{x}^i \frac{k! (-1)^{k-j} \bar{u}^{k-i} (\bar{v})^k}{i!(j-i)!(k-j)! (\bar{v})^i}, \quad 0 \leq j \leq k$$

$$\bar{u} = \frac{\gamma^2}{2 \alpha}, \quad \bar{v} = \frac{2 \beta}{\gamma^2}$$

$$(y)_0 = 1, \quad (y)_k = y(y+1) \cdots (y + k - 1), \quad k \geq 1.$$
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**Proof.** We have

\[
\prod_{m=1}^{k} b_m = k! \left[ \frac{\gamma^2}{2} k + \left( \beta - \frac{\gamma^2}{2} \right) \right] \cdots \left[ \frac{\gamma^2}{2} 1 + \left( \beta - \frac{\gamma^2}{2} \right) \right]
\]

\[
= k! \left( \frac{\gamma^2}{2} \right)^k \left[ \frac{2\beta}{\gamma^2} + k - 1 \right] \cdots \frac{2\beta}{\gamma^2}
\]

\[
= k! \left( \frac{\gamma^2}{2} \right)^k (\bar{v})_k.
\]

For \( i \leq j \),

\[
\prod_{\ell=i}^{j} (a_j - a_\ell) = \alpha^{k-i} \prod_{\ell=i}^{j-1} (j-\ell) \prod_{\ell=j+1}^{k} (j-\ell)
\]

\[
= (-1)^{k-j} \alpha^{k-i} (j-i)!(k-j)!.
\]

Hence

\[
\left( \prod_{m=i+1}^{k} b_m \right) \frac{1}{a_j - a_\ell} = \frac{k!}{i!} \left( \frac{\gamma^2}{2\alpha} \right)^{k-i} \frac{(\bar{v})_k}{(\bar{v})_i} \frac{(-1)^{k-j}}{(j-i)!(k-j)!}.
\]

The MGF of \( X_t \) will now be obtained by summing the moments of \( X_t \). We begin by assuming that the following sum converges:

\[
\sum_{k=0}^{\infty} s^k \mathbb{E} X_t^k = \sum_{j=0}^{\infty} e^{a_j t} \sum_{k=0}^{\infty} \frac{s^k}{k!} \theta_{kj} = \sum_{k=0}^{\infty} s^k \sum_{j=0}^{k} \frac{e^{a_j t}}{j!} \sum_{i=0}^{j} \mathbb{x}_i (-1)^{k-j-i} \frac{(\bar{v})_k}{(\bar{v})_i} \frac{(\bar{u})_i}{(\bar{v})_i} \frac{(\bar{u})_j}{(\bar{v})_i} (1 + s\bar{u})^{k-j}.
\]

This sum will be evaluated by summing first over \( k \), then over \( j \), and finally over \( i \). This procedure needs to be justified, as the summands are not all of the same sign. The justification is given below. We will use the formula

\[
(1 - y)^{-c} = \sum_{n=0}^{\infty} (c)_n \frac{y^n}{n!},
\]

which is valid for \( c \in \mathbb{R}, |y| < 1 \).

For fixed \( i \leq j \) and small enough \(|s|\), we have

\[
\sum_{k=j}^{\infty} \frac{\mathbb{x}_j}{k!} \frac{(-1)^{k-j} \bar{v}^{k-j}}{i!(j-i)!(k-j)!} \frac{(\bar{v})_k}{(\bar{v})_i} \frac{(\bar{u})_i}{(\bar{v})_i} (1 + s\bar{u})^{k-j}.
\]
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Next,

\[
\bar{x}^i \sum_{j=i}^{\infty} \frac{(\bar{v})_j s^j \bar{u}^j e^{\alpha j t}}{i!(j-i)!} (1 + s\bar{u})^{-\bar{v}-j} = s^i \bar{x}^i e^{\alpha it} \frac{(1 + s\bar{u})^{-\bar{v}-i}}{i!} \sum_{j=i}^{\infty} \frac{(\bar{v})_j s^{j-i} \bar{u}^{j-i} e^{\alpha (j-i)}}{(j-i)!} (1 + s\bar{u})^{-j-i} \\
= s^i \bar{x}^i e^{\alpha it} \frac{(1 + s\bar{u})^{-\bar{v}-i}}{i!} \left( 1 - \frac{s\bar{u} e^{\alpha t}}{1 + s\bar{u}} \right)^{-\bar{v}-i} \\
= s^i \bar{x}^i e^{\alpha it} \frac{1 - s\bar{u}(e^{\alpha t} - 1)}{i!}^{-\bar{v}-i}.
\]

Finally, summing over \(i\),

\[
\mathbb{E} e^{sX_t} = \left[ 1 - s\bar{u} (e^{\alpha t} - 1) \right]^{-\bar{v}} e^{s\bar{x} e^{\alpha t} \frac{1 - s\bar{u} (e^{\alpha t} - 1)}{1 - s\bar{u} (e^{\alpha t} - 1)}} = \phi(s)^\bar{v} e^{\lambda_t (\phi(s) - 1)}. \quad (2.15)
\]

Here

\[
\phi(s) = (1 - s\mu_t)^{-1}, \quad \mu_t = \frac{\gamma^2}{2} \left( \frac{e^{\alpha t} - 1}{\alpha} \right), \quad \lambda_t = \frac{2\alpha \bar{x}}{\gamma^2 (1 - e^{-\alpha t})}.
\]

The function \(\phi(s)\) is the MGF of an exponential distribution, with mean equal to \(\mu_t\). The distribution of \(X_t\) is thus the distribution of \(X'_t + X''_t\), where the two variables are independent, \(X'_t \sim \text{Gamma}(\bar{v}, \mu_t)\), and \(X''_t\) is compound Poisson

\[
X''_t = \sum_{i=1}^{N} U_i,
\]

with \(N \sim \text{Poisson}(\lambda_t)\), \(U_i \sim \text{Exp}(\mu_t)\).

We now show that the sum in (2.13) converges. What we have just done is to sum the quantities

\[
c_{ijk} = \frac{s^k \bar{x}^i (-1)^k \bar{u}^{k-i} e^{\alpha t} (\bar{v})_k}{i!(j-i)!(k-j)!}, \quad 0 \leq i \leq j \leq k < \infty.
\]

The same steps show that

\[
\sum_{0 \leq i \leq j \leq k < \infty} |c_{ijk}| = \left[ 1 - s\bar{u} (e^{\alpha t} + 1) \right]^{-\bar{v}} e^{-s\bar{x} e^{\alpha t} \frac{1 - s\bar{u} (e^{\alpha t} + 1)}{1 - s\bar{u} (e^{\alpha t} + 1)}},
\]

if \(0 \leq |s| < 1/[\bar{u} (e^{\alpha t} + 1)]\). Therefore: (1) the sum in (2.13) is convergent, (2) the order of summation of the \(c_{ijk}\) is irrelevant, and (3) the result is an analytic function of \(s\), at least for \(0 \leq |s| < 1/[\bar{u} (e^{\alpha t} + 1)]\). Expression (2.15) is therefore correct for \(s < 1/[\bar{u} (e^{\alpha t} - 1)]\), by analytic continuation.
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Obviously, the limit distribution of \( X_t \) as \( t \to \infty \) exists if, and only if, \( \alpha < 0 \), in which case it is \textbf{Gamma}(2/\beta, -\gamma^2/(2\alpha)) \) (the compound Poisson part, which depends on the initial condition \( \bar{x} \), disappears in the limit). Finally, the density of \( X_t \) may be obtained by conditioning on \( N \). To simplify the algebra, consider \( U = X_t/\mu_t \): with probability \( e^{-\lambda_1 \lambda_t^n/n!} \), the distribution of \( U \) is \textbf{Gamma}(\( \bar{v} + n \), 1), for \( n = 0, 1 \ldots \). Hence, the density of \( U \) is

\[
f_U(y) = \sum_{n=0}^{\infty} e^{-\lambda_1 \lambda_t^n} \frac{y^{\bar{v}+n-1}}{n! \Gamma(\bar{v}+n)} e^{-y} 1\{y>0\} = \left( \frac{y}{\lambda_t} \right)^{\frac{\bar{v}}{2}} e^{-\lambda_t-y} \sum_{n=0}^{\infty} \frac{(\sqrt{\lambda_t}y)^{\bar{v}+2n}}{n! \Gamma(\bar{v}+n)} 1\{y>0\}
\]

\[
= \left( \frac{y}{\lambda_t} \right)^{\frac{\bar{v}}{2}} e^{-\lambda_t-y} I_{\bar{v}-1}(2\sqrt{\lambda_t}y) 1\{y>0\},
\]

where \( I_{\nu} \) is the modified Bessel function of the first kind of argument \( \nu \) (Lebedev, 1972, p.108):

\[
I_{\nu}(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \quad \text{arg } z < \pi, \quad \nu \in \mathbb{C}. \quad (2.16)
\]

The density of \( X_t \) is thus

\[
f_{X_t}(x) = \frac{1}{\mu_t} \left( \frac{xe^{-\alpha t}}{\bar{x}} \right)^{\frac{\bar{v}}{2}} e^{-\lambda_t-x/\mu_t} I_{\bar{v}-1} \left( \frac{4\alpha}{\gamma^2(e^{\alpha t} - 1)} \sqrt{x\bar{x}e^{\alpha t}} \right) 1\{x>0\}. \quad (2.17)
\]

(When \( \bar{x} = 0 \) this reduces to the \textbf{Gamma}(\( \bar{v}, \mu_t \)) density.)

The MGF and the density above may also be obtained from the relationship between the solution of the SDE (1.1) and Bessel processes (see Revuz & Yor, 1999, Chapter 11). For a constant \( p > 0 \), the process \( \bar{X}_t = pX_t \) satisfies

\[
\bar{X}_t = p\bar{x} + \int_0^t (\alpha \bar{X}_s + \beta p) ds + \gamma \sqrt{p} \int_0^t \sqrt{\bar{X}_s} dW_s
\]

Letting \( p = 4/\gamma^2 \), we find that

\[
\bar{X}_t = \bar{x} + \int_0^t (\alpha \bar{X}_s + \bar{\beta}) ds + 2 \int_0^t \sqrt{\bar{X}_s} dW_s,
\]

where \( \bar{x} = 4\bar{x}/\gamma^2 \), \( \bar{\beta} = 4\beta/\gamma^2 \). Now, consider the square of a \( \delta \)-dimensional Bessel process, that is, the unique strong solution of

\[
Z_t = Z_0 + \delta t + 2 \int_0^t \sqrt{Z_s} dB_s
\]

(\( B \) standard Brownian motion), and let \( g(t) = (1 - e^{-\alpha t})/\alpha \) (see Revuz & Yor, 1999, Exercise (1.13), p.448). Integrating by parts, and using the time change formula for stochastic integrals (Revuz & Yor, 1999, p.180), we find

\[
e^{\alpha t} Z_{g(t)} = e^{\alpha t} \left[ Z_0 + \delta g(t) + 2 \int_0^t \sqrt{Z_{g(s)}} dB_{g(s)} \right]
\]

\[
= Z_0 + \alpha \int_0^t e^{\alpha s} Z_{g(s)} ds + \delta \int_0^t e^{\alpha s} dg(s) + 2 \int_0^t \sqrt{e^{\alpha s} Z_{g(s)} e^{\alpha s/2}} dB_{g(s)}.
\]
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The process

\[ \tilde{B}_t = \int_0^t e^{\alpha s/2} dB_g(s) \]

is a continuous martingale with quadratic variation equal to \( t \), and is thus standard Brownian motion (with respect to its own filtration). Hence, the process \( \tilde{Z}_t = e^{\alpha t} Z_{g(t)} \) satisfies

\[ \tilde{Z}_t = Z_0 + \int_0^t (\alpha \tilde{Z}_s + \delta) dt + 2 \int_0^t \sqrt{\tilde{Z}_s} d\tilde{B}_s. \]

Consequently, the processes \( \{\tilde{Z}_t; t \geq 0\} \) and \( \{\tilde{X}_t; t \geq 0\} \) have the same distribution, if

\[ Z_0 = \bar{x} = \frac{4\bar{x}}{\gamma^2}, \quad \delta = \beta = \frac{4\beta}{\gamma^2}. \]

Finally, the process \( \{X_t; t \geq 0\} \) has the same distribution as \( \{\gamma^2/4 e^{\alpha t} Z_{g(t)}; t \geq 0\} \). The MGF of \( Z_\tau \) is (Revuz & Yor, 1999, p.441)

\[ \mathbb{E} e^{\rho Z_\tau} = (1 - 2\rho \tau)^{-\delta/2} \exp \left[ \rho Z_0 / (1 - 2\rho \tau) \right]. \]

It can be checked that appropriate substitutions yield (2.15). In the same fashion, the density of \( Z_\tau \)

\[ f_{Z_\tau}(z) = \frac{1}{2\tau} \left( \frac{z}{Z_0} \right) \frac{1}{2}(\frac{\delta}{\gamma^2} - 1) e^{-(Z_0 + z)/2\tau} \sqrt{\frac{Z_0 z}{\tau}} I_{\frac{\delta}{\gamma^2} - 1} \left( \frac{\sqrt{Z_0 z}}{\tau} \right) \mathbf{1}_{\{z > 0\}}. \]

This implies that the density of \( X_t \) is (for \( x > 0 \))

\[
\frac{2\alpha e^{-\alpha t}}{\gamma^2 (1 - e^{-\alpha t})} \left( \frac{xe^{-\alpha t}}{x} \right)^{\frac{1}{2}(\frac{\delta}{\gamma^2} - 1)} \exp \left[ - \left( \frac{4\bar{x}}{\gamma^2} + \frac{4xe^{-\alpha t}}{\gamma^2} \right) \alpha / [2(1 - e^{-\alpha t})] \right] I_{\frac{\delta}{\gamma^2} - 1} \left( \frac{4\alpha \sqrt{x e^{-\alpha t}}}{\gamma^2 (1 - e^{-\alpha t})} \right)
\]

\[
= \frac{1}{\mu_t} \left( \frac{xe^{-\alpha t}}{x} \right)^{\frac{1}{2}(\frac{\delta}{\gamma^2} - 1)} e^{-\lambda_t - x / \mu_t} I_{\frac{\delta}{\gamma^2} - 1} \left( \frac{4\alpha \sqrt{x e^{-\alpha t}}}{\gamma^2 (1 - e^{-\alpha t})} \right),
\]

which agrees with (2.17).

It is also possible to find an expression for \( \mathbb{E} X_t^p \) for non-integral \( p \). One could integrate the density times \( x^p \) term by term; an alternative is the following: since

\[ \frac{1}{y^q} = \frac{1}{\Gamma(q)} \int_0^\infty s^{q-1} e^{-sy} ds, \quad q, y > 0, \quad (2.18) \]

we have

\[ \mathbb{E} X_t^{-q} = \frac{1}{\Gamma(q)} \int_0^\infty s^{q-1} e^{-sx} ds = \frac{1}{\Gamma(q)} \int_0^\infty s^{q-1} \phi(-s)^{\bar{v}} e^{\lambda_t (\phi(-s) - 1)} ds = \frac{\mu_t e^{-\lambda_t}}{\Gamma(q)} \int_0^1 r^{\bar{v}-q-1} (1-r)^{q-1} e^{\lambda_t r} dr \quad (2.19) \]

\[ = \mu_t^{-q} e^{-\lambda_t} \frac{\Gamma(\bar{v} - q)}{\Gamma(\bar{v})} {_1F_1}(\bar{v} - q, \bar{v}; \lambda_t), \quad (2.20) \]

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where \( \text{I}_1 \) is the confluent hypergeometric function

\[
\text{I}_1(r, s; z) = \sum_{n=0}^{\infty} \frac{(r)_n z^n}{(s)_n n!}, \quad r, s, z \in \mathbb{C}, \quad s \neq 0, -1, \ldots
\]

Here we have used the integral representation (9.11.1), p.266 of Lebedev (1972) for the confluent hypergeometric function. The above calculation is seen to be correct for \( 0 \leq q < \bar{v} \), while \( \mathbb{E}X_t^{-q} \) is infinite for \( q \leq \bar{v} \) (this is because the integral in (2.19) tends to infinity as \( q \uparrow \bar{v} \)). Since \( \mathbb{E}X_t^{-q} \) is an analytic function of \( q \), and (2.20) is analytic for \( \text{Re}(-q) > -\bar{v} \), we get the following result.

**Theorem 2.4.** \( \mathbb{E}X_t^p = \mu_t^p e^{-\lambda_t} \frac{\Gamma(\bar{v} + p)}{\Gamma(\bar{v})} \text{I}_1(\bar{v} + p, \bar{v}; \lambda_t), \quad p > -2\beta/\gamma^2. \)

This formula may be reconciled with the one in Theorem 2.3 by appealing to the identity (Lebedev, 1972, p.267)

\[
\text{I}_1(r, s; z) = e^z \text{I}_1(s-r, s; -z), \quad s \neq 0, -1, \ldots
\]

If \( p = k \), a non-negative integer, then \( \mathbb{E}X_t^k \) may be expressed as a series which terminates:

\[
\mathbb{E}X_t^k = \mu_t^k \frac{\Gamma(\bar{v} + k)}{\Gamma(\bar{v})} \text{I}_1(-k, \bar{v}; -\lambda_t)
\]

\[
= \sum_{i=0}^{k} \frac{(-1)^i(-k)_i(\bar{v})_i \mu_t^i \lambda_t^{k-i}}{i!} \mu_t^{k-i}
\]

\[
= \sum_{i=0}^{k} \frac{k!(\bar{v})_k}{(k-i)!(\bar{v})_i} \frac{\bar{x}^i e^{\alpha it}}{i!} \bar{u}^{k-i} \sum_{\ell=0}^{k-i} \binom{k-i}{\ell} e^{\alpha \ell t} (-1)^{k-i-\ell}
\]

\[
= \sum_{i=0}^{k} \frac{k!(\bar{v})_k}{(k-i)!(\bar{v})_i} \frac{\bar{x}^i e^{\alpha it}}{i!} \bar{u}^{k-i} \sum_{j=i}^{k} \frac{(k-i)!}{(j-i)!(k-j)!} e^{\alpha (j-i) t} (-1)^{k-j}
\]

\[
= \sum_{j=0}^{k} e^{\alpha jt} \sum_{i=0}^{j} \frac{\bar{x}^i k!(-1)^{k-j} \bar{u}^{k-i} (\bar{v})_k}{i!(j-i)!(k-j)!(\bar{v})_i}.
\]

3. Moments of the integrated square-root process

In this section, two techniques are given for the calculation of the moments of the integral of the square-root process \( \{Y_t\} \).

Suppose \( X \) is the solution of SDE (1.1), and define

\[
Y_t = \int_0^t X_s ds, \quad M_{jk}(t) = \mathbb{E}Y_t^j X_t^k,
\]
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for non-negative integers \( j,k \); \( M_{jk} \) is finite for all \( t \geq 0 \), since \( Y_t \) has finite moments of all order (since the Laplace transform of \( Y_t \) exists in a neighbourhood of the origin, see Section 4). Applying Ito’s formula, we get

\[
d(Y_t^jX_t^k) = X_t^k d(Y_t^j) + Y_t^j d(X_t^k)
\]

\[
= jY_t^{j-1}X_t^{k+1} dt + Y_t^j (a_k X_t^k + b_k X_t^{k-1}) dt + \gamma k Y_t^j X_t^{k-\frac{1}{2}} dW_t.
\]

Since \( E X_t^p < \infty \) for all positive \( p \), it follows that

\[
E \int_0^t \left( \gamma k Y_s^j X_s^{k-\frac{1}{2}} \right)^2 ds < \infty
\]

for all \( j,k \in \mathbb{N} \), and so

\[
E \int_0^t \gamma k Y_s^j X_s^{k-\frac{1}{2}} dW_s = 0.
\]

This implies

\[
\frac{d}{dt} M_{jk}(t) = a_k M_{jk}(t) + b_k M_{j,k-1}(t) + j M_{j-1,k+1}(t).
\]

(3.1)

These are linear ordinary differential equations with constant coefficients, and their solutions are straightforward (though tedious to obtain by hand). In order to find \( M_{j0}(t) \), it is necessary to calculate \( M_{j-1,1}(t), M_{j-2,2}(t), \) and so on. If we represent the \( \{ M_{jk} \} \) on a grid, with \( j \) horizontal and \( k \) vertical, \( M_{j0} \) is obtained after solving the differential equations on the diagonals \( M_{0,i}, \ldots, M_{i,0} \) for \( i \leq j \). The following result is proved by suitably adapting the arguments used in Section 2.

**Theorem 3.1.** Suppose \( \alpha \neq 0 \), and that \( X \) satisfies (1.1). Then, for \( j,k \in \mathbb{N} \),

\[
M_{jk}(t) = \sum_{m=0}^{j+k} M_{jkm}(t) e^{\alpha mt},
\]

(3.2)

where \( M_{jkm}(t) \) is a polynomial in \( t \) with degree

\[
\deg M_{jkm} \leq \begin{cases} j & \text{if } 0 \leq m \leq k \\ j + k - m & \text{if } k + 1 \leq m \leq j + k. \end{cases}
\]

(3.3)

**Proof.** We proceed by induction on \( j \) and then on \( k \). The theorem is correct for \( j = 0 \) and all \( k \geq 0 \) by Theorem 2.3. Suppose (3.2)-(3.3) are correct for \( j = 0, \ldots, J-1 \) and all \( k \geq 0 \), for some \( J \geq 1 \). Based on this assumption, the solution of (3.1) is

\[
M_{Jk} = e^{\alpha_k t} M_{Jk}(0) + e^{\alpha_k t} \int_0^t e^{-\alpha_k s} R_{Jk}(s) ds,
\]

(3.4)

where

\[
R_{Jk}(t) = b_k M_{J,k-1}(t) + JM_{J-1,k+1}(t)
\]

\[
= \sum_{m=0}^{J+k-1} [b_k M_{J,k-1,m}(t) + JM_{J-1,k+1,m}(t)] e^{\alpha_m t} + JM_{J-1,k+1,J+k}(t) e^{\alpha_{J+k} t}.
\]
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We show that this implies that (3.2)-(3.3) hold for $j = J, k = 0$. Since $a_0 = b_0 = M_{J0}(0) = 0$, (3.4) reduces to

$$M_{J0}(t) = J \sum_{m=0}^{J-1} \int_0^t M_{J-1,1,m}(s)e^{a_ms} ds. \quad (3.5)$$

Consider the terms corresponding to $m = 1, \ldots, J$ in the sum above. Recall that $a_m = \alpha m \neq 0$ and that, for any constant $A \neq 0$,

$$\int_0^t e^{As} s^n ds$$

is the sum of a constant and of a polynomial of degree $n$ times $e^{At}$. Hence

$$\int_0^t M_{J-1,1,m}(s)e^{a_ms} ds$$

is the sum of a constant and of a polynomial of the same degree as $M_{J-1,1,m}$ times $e^{a_mt}$.

Since $a_0 = 0$, the first term on the right hand side of (3.5) is seen to be a polynomial of degree equal to one plus the degree of $M_{J-1,1,0}$.

The above considerations imply that

$$M_{J0}(t) = \sum_{m=0}^{J-1} M_{J0m}(t)e^{\alpha mt},$$

where the functions $M_{J0m}(t)$ are polynomials in $t$. Moreover, the induction assumption says in particular that

$$\deg M_{J-1,1,m}(t) \leq \begin{cases} J-1 & \text{if } 0 \leq m \leq 1 \\ J-m & \text{if } 2 \leq m \leq J \end{cases}$$

and, consequently, what we have just seen also means that

$$\deg M_{J0m} \leq \begin{cases} J & \text{if } m = 0 \\ J-m & \text{if } 1 \leq m \leq J. \end{cases}$$

Keeping $J$ fixed, make another induction assumption: (3.2)-(3.3) hold for $j = J$ and $k = 0, \ldots, K-1$, for some $K \geq 1$. The proof will be finished once we show that (3.2)-(3.3) hold for $j = J$ and $k = K$ as well. By the induction assumptions and (3.4),

$$M_{JK}(t) = e^{a_Kt} \left\{ \sum_{m=0}^{J+K-1} \int_0^t \left[ b_k M_{J,k-1,m}(s) + JM_{J-1,K+1,m}(s) \right] e^{(am-a_K)s} ds \right\} + \int_0^t JM_{J-1,K+1,J+K}(s)e^{(aJ+K-a_K)s} ds \quad (3.6)$$

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By the same reasoning as before, we see that $M_{JK}$ is of the form

$$M_{JK}(t) = \sum_{m=0}^{J+K} M_{JKm}(t)e^{a_m t},$$

where $M_{JKm}(t)$ is a polynomial. In (3.6), the terms of the sum inside the curly brackets with $m \neq K$ reduce to

$$\text{constant + polynomial } \times e^{(a_m - a_K)t}$$

The degree of the polynomial is the same as that of

$$b_k M_{J,K-1,m}(s) + J M_{J-1,K+1,m}(s),$$

that is, no larger than

$$\max[\min(J, J + K - 1 - m), \min(J - 1, J + K - m)] = \min(J, J + K - m). \quad (3.7)$$

The same applies to the term following the sum, which corresponds to $m = J + K$. For the term $m = K$ inside the curly brackets, the integral reduces to a polynomial of degree no larger than

$$1 + \max[\deg(M_{J,K-1,K}), \deg(M_{J-1,K+1,K})] = 1 + \max(J - 1, J - 1) = J. \quad (3.8)$$

All these integrals are then multiplied by $e^{a_K t}$, so that (3.7) becomes the degree of $M_{JKm}$ for $m \neq K$, and (3.8) becomes the degree of $M_{JKK}$. This ends the proof.

A general simplified formula for the polynomials $M_{jkm}$ has not been found, but they can be calculated recursively, as the next theorem shows.

**Theorem 3.2.** If we let

$$M_{jkm}(t) = \sum_{n=0}^{j \wedge (j+k-m)} M_{jkmn} t^n,$$

$$R_{jkmn} = b_k M_{j,k-1,m,n} + j M_{j-1,k+1,m,n},$$

then

$$M_{jkmn} = -\sum_{i=n}^{j \wedge (j+k-m)} \frac{(n+1)i-n}{[\alpha(k-m)]^i-n+1} R_{jkmn}, \quad k \neq m \quad (3.9)$$

$$M_{jkkn} = -\frac{1}{n} R_{j,k,k,n-1}, \quad n = 1, \ldots, j \quad (3.10)$$

$$M_{jko0} = -\sum_{m=0}^{j+k} M_{jkm0}, \quad j \geq 1 \quad (3.11)$$

$$M_{0km0} = \theta_{km} \quad \forall (k,m) \quad (\text{see Theorem 2.3}) \quad (3.12)$$

$$M_{0kmn} = 0, \quad n \geq 1. \quad (3.13)$$
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**Proof.** By Theorem 3.1, the derivative on the left hand side of Eq.(3.1) equals

\[
\frac{d}{dt} \sum_{m=0}^{j+k} e^{a_m t} \sum_{n=0}^{j} M_{jkmn} t^n = \sum_{m=0}^{j+k} e^{a_m t} \sum_{n=0}^{j} [a_m M_{jkmn} + (n+1)M_{j,k,m,n+1}] t^n.
\]

Here, in order to simplify the notation, we have added some \(M_{jkmn}\) which are necessarily 0, since \(n > \min(j, j + k - m)\). The right hand side of (3.1) may be written as

\[
\sum_{m=0}^{j+k} e^{a_m t} \sum_{n=0}^{j} [a_k M_{jkmn} + b_k M_{j,k-1,m,n} + jM_{j-1,k+1,m,n}] t^n.
\]

Hence,

\[
M_{j,k,m,n+1} = \frac{1}{n+1} [(a_k - a_m)M_{jkmn} + R_{jkmn}].
\] (3.14)

If \(k = m\), then this is (3.10). If \(k \neq m\), it can be verified that (3.9) is the solution of (3.14) that satisfies the condition

\[
M_{j,k,m,1+j\wedge(j+k-m)} = 0
\]

(which must hold by Theorem 3.1). Eq.(3.11) results from \(M_{jk}(0) = 0\) for any \(j \geq 1\), while (3.12)-(3.13) follow from Theorem 2.3.

Eqs.(3.9)-(3.12) are easy to program, and give the explicit formulas for any moment of \(Y_t\) much faster than solving the differential equations (3.1). The recursion must proceed through the values \((j, k) = (0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0)\), and so on, up to the highest required moment \(EY_t^J = M_{J0}(t)\). Observe that those formulas only need to be generated symbolically once. The first three moments of \(Y_t\) are:

\[
EY_t = -\frac{x}{\alpha} - \frac{\beta}{\alpha} - \frac{t\beta}{\alpha} + e^{t\alpha} \left( \frac{x}{\alpha} + \frac{\beta}{\alpha^2} \right)
\]

\[
EY_t^2 = \frac{\bar{x}^2}{\alpha^2} + \frac{2\bar{x}\beta}{\alpha^3} - \frac{\beta^2}{\alpha^4} - \frac{\bar{\gamma}^2}{\alpha^5} - \frac{5\beta\gamma}{2\alpha^6} + t \left( \frac{2\bar{x}\beta}{\alpha^2} + \frac{2\beta^2}{\alpha^3} - \frac{\beta\gamma}{\alpha^4} \right) + \frac{t^2\beta^2}{\alpha^2} + e^{t\alpha} \left[ -\frac{2\bar{x}^2}{\alpha^2} - \frac{4\bar{x}\beta}{\alpha^3} + \frac{2\beta^2}{\alpha^4} + \frac{2\bar{\gamma}^2}{\alpha^5} + t \left( -\frac{2\bar{x}\beta}{\alpha^2} - \frac{2\beta^2}{\alpha^3} + \frac{2\bar{\gamma}^2}{\alpha^4} - \frac{2\beta\gamma}{\alpha^5} \right) \right]
\]

\[
EY_t^3 = \frac{-\bar{x}^3}{\alpha^3} + 3\frac{\bar{x}^2\beta}{\alpha^4} - \frac{3\bar{x}\beta^2}{\alpha^5} + \frac{\beta^3}{\alpha^6} + \frac{3\bar{x}^2\gamma}{\alpha^4} + \frac{21\bar{x}\beta\gamma}{2\alpha^5} + \frac{15\beta^2\gamma}{2\alpha^6} - \frac{3\bar{x}\gamma^2}{\alpha^5} - \frac{11\beta\gamma^2}{\alpha^6}
\]

\[
+ t \left( -\frac{3\bar{x}^2\beta}{\alpha^3} + \frac{6\bar{x}\beta^2}{\alpha^4} - \frac{3\beta^3}{\alpha^5} + \frac{6\bar{x}\beta\gamma}{\alpha^4} + \frac{21\beta^2\gamma}{2\alpha^5} - \frac{3\beta\gamma^2}{\alpha^6} \right)
\]

\[
+ t^2 \left( -\frac{3\bar{x}\beta^2}{\alpha^3} - \frac{3\beta^3}{\alpha^4} + \frac{3\beta^2\gamma^2}{\alpha^5} \right) - \frac{t^3\beta^3}{\alpha^3}
\]

\[
+ e^{t\alpha} \left[ \frac{3\bar{x}^3}{\alpha^3} + 9\frac{\bar{x}^2\beta}{\alpha^4} + 9\frac{\bar{x}\beta^2}{\alpha^5} + \frac{3\beta^3}{\alpha^6} - \frac{3\bar{x}^2\gamma}{\alpha^4} - \frac{33\bar{x}\beta\gamma}{2\alpha^5} - \frac{27\beta^2\gamma}{2\alpha^6} - \frac{3\bar{x}\gamma^2}{2\alpha^5} + \frac{15\beta\gamma^2}{2\alpha^6} \right]
\]
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\[ + t \left( \frac{6\bar{x}^2\beta}{\alpha^3} + \frac{12\bar{x}^2\beta}{\alpha^4} + \frac{6\bar{x}^2\gamma^2}{\alpha^3} + \frac{9\bar{x}\beta\gamma}{\alpha^4} - \frac{3\beta^2\gamma^2}{\alpha^5} - \frac{3\bar{x}\gamma^4}{\alpha^4} - \frac{9\beta\gamma^4}{\alpha^5} \right) \\
+ t^2 \left( \frac{3\bar{x}^2\beta^2}{\alpha^3} + \frac{6\bar{x}^2\beta\gamma}{\alpha^4} + \frac{6\bar{x}^2\gamma^2}{\alpha^3} + \frac{3\bar{x}\gamma^4}{\alpha^4} + \frac{3\beta\gamma^4}{\alpha^5} \right) \\
+ e^{2\alpha t} \left[ - \frac{3\bar{x}^3}{\alpha^3} - \frac{9\bar{x}^2\beta}{\alpha^4} + \frac{9\bar{x}^2\beta}{\alpha^5} - \frac{3\bar{x}^2\gamma^2}{\alpha^4} + \frac{3\beta\gamma^2}{\alpha^5} + \frac{9\beta^2\gamma^2}{\alpha^5} + \frac{3\bar{x}\gamma^4}{\alpha^4} + \frac{3\beta\gamma^4}{\alpha^5} \right] \\
+ e^{3\alpha t} \left( \frac{\bar{x}^3}{\alpha^3} + \frac{3\bar{x}^2\beta}{\alpha^4} + \frac{3\bar{x}^2\gamma^2}{\alpha^3} + \frac{3\bar{x}^2\gamma^2}{\alpha^4} + \frac{9\bar{x}\beta\gamma}{\alpha^5} + \frac{3\beta^2\gamma^2}{\alpha^5} + \frac{3\bar{x}\gamma^4}{\alpha^4} + \frac{\beta\gamma^4}{\alpha^5} \right). \]

4. Laplace transform, inverse moments and density of $Y_t$

In this section and the next one, the symbol $\mathcal{O}$ (“big oh”) has the following meaning:
we write

\[ g(s) = \mathcal{O}(h(s)) \quad \text{as} \quad s \to \infty \]

if \(|g(s)/h(s)|\) remains bounded as \(s \to \infty\). In words, \(g(s)\) is big oh of \(h(s)\) as \(s\) tends to
infinity if \(|g(s)|\) is no larger than some constant times \(|h(s)|\) when \(s\) is large. We write

\[ g(s) = o(h(s)) \quad \text{as} \quad s \to \infty \]

(“\(g(s)\) is little oh of \(h(s)\) as \(s \to \infty\)”)

if

\[ \lim_{s \to \infty} \frac{g(s)}{h(s)} = 0. \]

Another symbol used below is “\(\sim\)”, meaning “asymptotically equal to”: we write \(g(s) \sim h(s)\) if

\[ \lim_{s \to \infty} \frac{g(s)}{h(s)} = 1. \]

Finally, “\(i\)” is now the imaginary unit of the complex numbers (that is, \(i^2 = -1\)).

Since the Laplace transform of the integral of the squared Bessel process may be
derived explicitly (see Revuz & Yor, 1999, p.445), it is not surprising that the Laplace
transform of the square-root process also has an explicit expression. A quick way to find
\(\mathbb{E} \exp(-sY_t)\) is to suitably modify the CIR formula for the price of a zero-coupon bond
(Cox et al., 1985). Suppose the short rate follows a process of type (1.1); the bond price
is known to be the expectation of the exponential of minus the integral of the short rate.
Multiplying the spot rate by a positive number yields another process which also satisfies
(1.1), but with different parameters. Rewriting the bond price formula for this new process
immediately yields the Laplace transform of $Y_t$. The details follow.

Suppose $X$ is the solution of (1.1), let $s > 0$, and define

\[ \tilde{X}_t = sX_t, \quad \tilde{X}_0 = \bar{x} = s\bar{x}. \]
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Then

$$\tilde{X}_t = \tilde{x} + \int_0^t (\alpha \tilde{X}_u + \tilde{\beta}) \, du + \int_0^t \tilde{\gamma} \sqrt{\tilde{X}_u} \, dW_u,$$

where $\tilde{\beta} = \beta s$ and $\tilde{\gamma} = \gamma \sqrt{s}$. The formula for the price of a zero-coupon bond maturing in $t$ years is (Cox et al., 1985)

$$E e^{-s Y_t} = E e^{-\int_0^t \tilde{X}_u \, du} \left[ \frac{\sqrt{\alpha^2 + 2\tilde{\gamma}^2} e^{(\sqrt{\alpha^2 + 2\tilde{\gamma}^2} - \alpha) t/2}}{(\sqrt{\alpha^2 + 2\tilde{\gamma}^2} - \alpha)(e^{\sqrt{\alpha^2 + 2\tilde{\gamma}^2} t} - 1) + 2\sqrt{\alpha^2 + 2\tilde{\gamma}^2}} \right]^{2\tilde{\gamma}/\sqrt{\alpha}} \times \exp \left[ -\tilde{x} \frac{2(e^{\sqrt{\alpha^2 + 2\tilde{\gamma}^2} t} - 1)}{(\sqrt{\alpha^2 + 2\tilde{\gamma}^2} - \alpha)(e^{\sqrt{\alpha^2 + 2\tilde{\gamma}^2} t} - 1) + 2\sqrt{\alpha^2 + 2\tilde{\gamma}^2}} \right] \right]^{2\tilde{\beta}/\sqrt{\alpha}} \exp \left[ -\tilde{x} \frac{2\sinh(P t/2)}{P \cosh(P t/2) - \frac{s}{P} \sinh(P t/2)} \right] \right], \quad (4.1)$$

where $P = P(s) = \sqrt{\alpha^2 + 2\tilde{\gamma}^2} s$.

The Laplace transform of $Y_t$ is finite in a neighborhood of the origin. This is seen by noting that

$$\cosh(P t/2), \quad \frac{\alpha}{P} \sinh(P t/2)$$

are both analytic functions of $P$, and that their difference does not vanish at $P(0) = |\alpha|$, for any values of the parameters $\alpha, \beta, \gamma$ considered, and for any $t > 0$. Consequently, (1) all the moments of $Y_t$ are finite, and (2) they are given by the familiar formula

$$E Y_t^k = (-1)^n \frac{d^n}{ds^n} E e^{-s Y_t} \bigg|_{s=0}, \quad k \in \mathbb{N}.$$

The Laplace transform gives a third method of calculating the moments of $Y_t$. The author has compared the computation times for the moments of $Y_t$ up to order 6 using Mathematica; the recursive method of Section 3 does a lot better than the others, especially for higher moments. As an example, on what is now a rather slow machine (Macintosh 3400 180 Mhz), Mathematica took 377 seconds to find the general symbolic formula for the 6th moment by differentiating the Laplace transform, and a bit more to find the general formulas for moments of order 1 to 6 by solving differential equations (3.1). The recursive method produced all 6 moments in a little more than 8 seconds. A program recursively calculating just the numerical values of the moments (rather than the general symbolic formulas) would of course execute much faster, especially in C.

The next theorem concerns the moments and MGF of $1/Y_t$.

**Theorem 4.1.** (a) $E Y_t^r$ is finite for all $r \in \mathbb{R}$, and

$$E Y_t^{-q} = \frac{1}{\Gamma(q)} \int_0^\infty s^{q-1} E \exp(-s Y_t) \, ds, \quad q > 0.$$
The integrated square-root process

(b) \( \mathbb{E} \exp \left( \frac{p}{Y_t} \right) = 1 + \sqrt{p} \int_0^\infty s^{-1/2} I_1(2\sqrt{ps}) \mathbb{E} e^{-s Y_t} \, ds \) for all \( p \geq 0 \).

This is finite if, and only if, \( 0 \leq p < \frac{1}{2\gamma^2} (\beta t + \bar{x})^2 \).

(c) \( \mathbb{E} \exp \left( \frac{p}{X_s} \right) = \infty \) for all \( p \geq \frac{1}{2\gamma^2} (\beta + \bar{x})^2 \).

**Proof.** By (2.14),

\[
P = \gamma \sqrt{2s} \sqrt{1 + \frac{\alpha^2}{2\gamma^2 s}} = \gamma \sqrt{2s} \left[ 1 + \mathcal{O} (s^{-1}) \right].
\]

From this, we can find the asymptotic behaviour of the first factor in (4.1):

\[
\left\{ \frac{e^{-\alpha t/2}}{\cosh(Pt/2) - \frac{\alpha}{P} \sinh(Pt/2)} \right\}^{2\beta/\gamma^2} = 2^{\frac{2\beta}{\gamma^2}} e^{-\alpha \beta t/\gamma^2} \left[ \left( 1 - \frac{\alpha}{P} \right) e^{Pt/2} + \left( 1 + \frac{\alpha}{P} \right) e^{-Pt/2} \right]^{-\frac{2\beta}{\gamma^2}}
\]

\[
= 2^{\frac{2\beta}{\gamma^2}} e^{-\alpha \beta t/\gamma^2 - \beta Pt/\gamma^2} \left[ \left( 1 - \frac{\alpha}{P} \right) + \left( 1 + \frac{\alpha}{P} \right) e^{-Pt} \right]^{-\frac{2\beta}{\gamma^2}}
\]

\[
\sim 2^{\frac{2\beta}{\gamma^2}} e^{-\alpha \beta t/\gamma^2 - \beta t \sqrt{2s}/\gamma}
\]

as \( s \) tends to infinity. Now turn to the second factor:

\[
- \frac{s \bar{x}}{P} \frac{2 \sinh(Pt/2)}{\cosh(Pt/2) - \frac{\alpha}{P} \sinh(Pt/2)} = - \frac{2s \bar{x}}{\sqrt{\alpha^2 + 2\gamma^2 s}} \frac{1}{\text{coth}(Pt/2) - \frac{\alpha}{\sqrt{\alpha^2 + 2\gamma^2 s}}}
\]

\[
= - \sqrt{2s \bar{x}} \gamma \left( 1 + \frac{\alpha^2}{2\gamma^2 s} \right)^{-\frac{1}{2}} \frac{1}{\frac{1+e^{-Pt}}{1-e^{-Pt}} - \frac{\alpha}{\gamma \sqrt{2s} \left( 1 + \frac{\alpha^2}{2\gamma^2 s} \right)^{\frac{1}{2}}}}
\]

\[
= - \sqrt{2s \bar{x}} \gamma \left[ 1 + \mathcal{O} (s^{-1}) \right] \frac{1}{1 + \mathcal{O}(e^{-Pt}) - \frac{\alpha}{\gamma \sqrt{2s}} + \mathcal{O} \left( s^{-\frac{3}{2}} \right)}
\]

\[
= - \sqrt{2s \bar{x}} \gamma \left[ 1 + \mathcal{O} (s^{-1}) \right] \left[ 1 + \frac{\alpha}{\gamma \sqrt{2s}} + \mathcal{O} (s^{-1}) \right]
\]

\[
= - \sqrt{2s \bar{x}} \gamma - \frac{\alpha \bar{x}}{\gamma^2} + \mathcal{O} \left( s^{-\frac{3}{2}} \right).
\]

From these expressions we get

\[
\mathbb{E} e^{-s Y_t} \sim 2^{\frac{2\beta}{\gamma^2}} \exp \left\{ - (\beta t + \bar{x}) \left( \frac{\alpha}{\gamma^2} + \frac{\sqrt{2s}}{\gamma} \right) \right\} \quad \text{as} \quad s \to \infty. \quad (4.2)
\]
The integrated square-root process

By this asymptotic identity and (2.18), EY_t^{-q} is finite if, and only if, the integral

\[ 2^{\frac{q}{2}} e^{-\frac{1}{2}(\beta t + \bar{x})} \frac{1}{\Gamma(q)} \int_{0}^{\infty} s^{q-1} \exp \left\{ -\frac{\sqrt{2s}}{\gamma} (\beta t + \bar{x}) \right\} ds, \]

is finite, which is true for all q > 0. This proves part (a) of the theorem.

Now turn to part (b). For p \geq 0,

\[ \mathbb{E} \left( \exp \left( \frac{p}{Y_t} \right) \right) = \sum_{n=0}^{\infty} \frac{p^n}{n!} \mathbb{E}Y_t^{-n} \]

\[ = 1 + \int_{0}^{\infty} \sum_{n=1}^{\infty} \frac{p^n s^{n-1}}{\Gamma(n)\Gamma(n+1)} \mathbb{E}(-sY_t) ds \]

\[ = 1 + \int_{0}^{\infty} \sqrt{\frac{p}{s}} \sum_{k=0}^{\infty} \frac{(2\sqrt{ps}/2)^{1+2k}}{\Gamma(k+1)\Gamma(k+2)} \mathbb{E}(-sY_t) ds \]

\[ = 1 + \int_{0}^{\infty} \sqrt{\frac{p}{s}} I_1(2\sqrt{ps}) \mathbb{E}(-sY_t) ds \]

(see the definition of the modified Bessel function I_\nu in (2.16)).

It remains to be seen for what p > 0 the above integral is finite. The integrand is continuous in s, and bounded near the origin. It is known that

\[ I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}, \quad x \to \infty \]  

(Lebedev, 1972, p.123). Combining this with (4.2), we find that

\[ \sqrt{\frac{p}{s}} I_1(2\sqrt{ps}) \mathbb{E}(-sY_t) \sim Cs^{-3/4} \exp \left\{ \sqrt{2s} \left[ \sqrt{2p} - \frac{1}{\gamma} (\beta t + \bar{x}) \right] \right\} \]

as s \to \infty (here C > 0 depends on p but not on s). The function on the right is integrable over [1, \infty) if, and only if,

\[ p < \frac{1}{2\gamma^2 (\beta t + \bar{x})^2}. \]

To prove (c), note that by the Cauchy-Schwarz inequality

\[ \left[ \int_{0}^{t} d\tau \right]^{2} \leq \int_{0}^{t} \frac{d\tau}{X_\tau} \int_{0}^{t} X_\tau d\tau, \]

and thus

\[ p \int_{0}^{t} \frac{d\tau}{X_\tau} \geq \frac{pt^2}{\int_{0}^{t} X_\tau d\tau}. \]
The result follows from (b).

By contrast, it is easy to see that the density of $X_t$ behaves like $x^{\bar{v}-1}$ as $x \downarrow 0$, with $\bar{v} = 2\beta/\gamma^2$ (see (2.17) and Theorem 2.4), and thus $E X_t^p < \infty$ for $p > -\bar{v}$ only, whatever the initial condition $\bar{x} \geq 0$, and, consequently, $E \exp(q/X_t) = \infty$ for any $q > 0$. Thus, the density of $Y_t$ tends to 0 much quicker than that of $X_t$, as the argument tends to 0. This is a little unexpected, especially since $Y_0 = 0$ with probability one, irrespective of $X_0 = \bar{x}$. An intuitive explanation is that $Y_t$ is close to 0 if $X_s$ is close to 0 for “many” $s$ between 0 and $t$, which is less likely than $X_s$ being close to 0 for one particular $s$; integration is therefore a “smoothing” operation in this case. This relates to Theorem 4.2 below, which says that the density of $Y_t$ is “infinitely flat” at the origin (that is, all its derivatives vanish).

Part (c) of the theorem is related to absence of arbitrage and changes of measures in the financial model

$$dS = \mu S dt + \sqrt{X} S dV,$$

where $X$ satisfies (1.1) and $(V, W)$ is two-dimensional Brownian motion (possibly correlated). If one attempts the usual Girsanov change of measure to make $\{ e^{-rt} S_t \}$ a martingale, then one has the problem of checking that the stochastic exponential of minus the integral of the market price of risk

$$\frac{\mu - r}{\sqrt{X}}$$

has expectation equal to one (which would make it a martingale, see Karatzas & Shreve (1998), p.12). A sufficient condition for this to hold is Novikov’s condition

$$E \exp \left( \frac{1}{2} \int_0^t \frac{(\mu - r)^2}{X_s} ds \right) < \infty.$$ 

Theorem 4.1 shows that Novikov’s condition fails if

$$(\mu - r)^2 \geq \frac{1}{\gamma^2} \left( \beta + \frac{\bar{x}}{t} \right)^2$$

(but does not say that the condition holds otherwise).

Observe that, by Theorem 2.4, $E(1/X_s) = \infty$ for all $s > 0$ if $2\beta/\gamma^2 \leq 1$, which implies

$$E \int_0^t \frac{1}{X_s} ds = \infty,$$

and thus $E e^p \int_0^t ds/X_s = \infty$ for all $p > 0$. More generally, since $E X_t^{-q} = \infty$ for all $q \geq 2\beta/\gamma^2$, it is plausible that not all moments of $\int_0^t \frac{1}{X_s} ds$ are finite (though the author has not been able to prove this), which would then imply $E e^p \int_0^t ds/X_s = \infty$ for all $p > 0$, for any $\beta, \gamma$ and $t > 0$. 

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Theorem 4.2. Suppose $t > 0$ and let $g_t(\cdot)$ be the density of $Y_t$ with respect to Lebesgue measure on $\mathbb{R}$. Then $g_t(x)$ exists and is a continuous, uniformly bounded function of $x \in \mathbb{R}$. The same applies to $(d^n/dx^n)g_t(x)$, for any $n \geq 1$.

Proof. Apply the following well-known property of characteristic functions (see Theorem 3 and Lemma 2 in Feller (1971), pp.509 and 512):

Suppose a r.v. $U$ satisfies

$$\int_{-\infty}^{\infty} |Ee^{i\zeta U} \zeta^n| d\zeta < \infty$$

for some $n \geq 1$; then the distribution of $U$ has a density which is everywhere continuous and uniformly bounded, and the $n$th derivative of this density is also everywhere continuous and uniformly bounded.

The characteristic function of $Y_t$ is

$$\psi(\zeta) = Ee^{i\zeta Y_t} = Ee^{-sY_t} \bigg|_{s=-i\zeta}$$

$$= \left[ \frac{e^{-\alpha t/2}}{\cosh(Zt/2) - \frac{\alpha}{2} \sinh(Zt/2)} \right]^{\frac{2\alpha}{\gamma^2}} \exp \left[ \frac{i\zeta \bar{x}}{Z} \frac{2 \sinh(Zt/2)}{\cosh(Zt/2) - \frac{\alpha}{2} \sinh(Zt/2)} \right],$$

where $Z = Z(\zeta) = \sqrt{\alpha^2 - 2\gamma^2 \zeta i}$. First, consider $\psi(\zeta)$ as $\zeta \to \infty$. We have (see (2.14))

$$Z(\zeta) - \gamma(1-i)\sqrt{\zeta} = \gamma(1-i)\sqrt{\zeta} \left( \sqrt{1 - \frac{\alpha^2}{2\gamma^2 \zeta i}} - 1 \right)$$

$$= \gamma(1-i)\sqrt{\zeta} \sum_{k=1}^{\infty} \left( \frac{\alpha^2}{2\gamma^2 \zeta i} \right)^{k} \frac{(-1)^k}{k!}$$

$$\to 0$$

as $\zeta \to \infty$. This implies

$$e^{Zt/2} \sim e^{\gamma(1-i)\sqrt{\zeta}t/2}$$

$$\cosh(Zt/2) - \frac{\alpha}{Z} \sinh(Zt/2) \sim \frac{1}{2} e^{\gamma(1-i)\sqrt{\zeta}t/2}.$$

Moreover,

$$\frac{i\zeta \bar{x}}{Z} \frac{2 \sinh(Zt/2)}{\cosh(Zt/2) - \frac{\alpha}{2} \sinh(Zt/2)} = \frac{2i\zeta \bar{x}}{Z} \frac{1}{\coth(Zt/2) - \frac{\alpha}{Z}}$$

$$= \frac{(-1+i)\bar{x}\sqrt{\zeta}}{\gamma} \left( 1 - \frac{\alpha^2}{2\gamma^2 \zeta i} \right)^{-\frac{1}{2}} \frac{1}{\frac{1+e^{-Zt}}{1-e^{-Zt}} - \frac{\alpha}{\gamma(1-i)\sqrt{\zeta}} \left( 1 - \frac{\alpha^2}{2\gamma^2 \zeta i} \right)^{-\frac{1}{2}}}$$
The integrated square-root process

\[
\begin{align*}
= \frac{(-1 + i)x\sqrt{\zeta}}{\gamma} (1 + \mathcal{O}(\zeta^{-1})) \left(1 + \mathcal{O}(e^{-Zt}) - \frac{1}{\gamma(1 - i)\sqrt{\zeta}}(1 + \mathcal{O}(\zeta^{-1}))\right) \\
= \frac{(-1 + i)x\sqrt{\zeta}}{\gamma} (1 + \mathcal{O}(\zeta^{-1})) \left(1 + \frac{\alpha}{\gamma(1 - i)\sqrt{\zeta}} + \mathcal{O}(\zeta^{-1})\right) \\
= \frac{(-1 + i)x\sqrt{\zeta}}{\gamma} \alpha \bar{x} + \mathcal{O}(\zeta^{-\frac{1}{2}})
\end{align*}
\]

Putting all this together, we find

\[|\psi(\zeta)| \sim C_1 e^{-C_2 \sqrt{\zeta}}\]

as \(\zeta \to \infty\), where \(C_1, C_2 > 0\) do not depend on \(\zeta\). Hence

\[
\int_0^{\infty} |\psi(\zeta)\zeta^n| d\zeta < \infty
\]

for any \(n \geq 0\). The integral from \(-\infty\) to 0 is handled similarly: as \(\zeta \to -\infty\),

\[
Z(\zeta) - \gamma(1 + i)\sqrt{-\zeta} \to 0,
\]

\[e^{Zt/2} \sim e^{\gamma(1 + i)\sqrt{-\zeta}t/2}\]

\[\cosh(Zt/2) - \frac{\alpha}{Z} \sinh(Zt/2) \sim \frac{1}{2} e^{\gamma(1 + i)\sqrt{-\zeta}t/2}\]

\[
\frac{i\zeta \bar{x}}{Z} \frac{2\sinh(Zt/2)}{\cosh(Zt/2) - \frac{\alpha}{Z} \sinh(Zt/2)} = \frac{(-1 + i)x\sqrt{-\zeta}}{\gamma} - \frac{\alpha \bar{x}}{\gamma^2} + \mathcal{O}((-\zeta)^{-\frac{1}{2}})
\]

\[|\psi(\zeta)| \sim C_3 e^{-C_4 \sqrt{-\zeta}}.\]

5. Relationships between the Laplace transforms of \(U\) and \(1/U\)

The formula in Theorem 4.1 (b) applies for any random variable \(U > 0\). The same proof leads to the identity

\[
E e^{pU} = 1 + \int_0^\infty \sqrt{\frac{p}{s}} I_1(2\sqrt{ps}) \exp(-sU) ds, \quad p \geq 0 \quad (5.1)
\]

(both sides are simultaneously finite or infinite, by Fubini’s Theorem). Formula (5.1) strangely resembles a known relationship for Laplace transforms (Oberhettinger & Badii, 1973, p.4):

\[
\int_0^\infty e^{-pt}t^{\nu-1}f(t^{-1}) dt = \int_0^\infty \left(\frac{s}{p}\right)^\nu J_\nu(2\sqrt{ps}) \left[\int_0^\infty e^{-st}f(t) dt\right] ds, \quad \nu > -1. \quad (5.2)
\]

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Here, \( J_\nu \) is the ordinary Bessel function of order \( \nu \) (Lebedev, 1972, p.102),

\[
J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \quad |\arg z| < \pi, \quad \nu \in \mathbb{C}.
\]

Let us rephrase (5.2) in probabilistic terms, by considering that \( f(\cdot) \) is the density of a random variable \( U > 0 \):

\[
\mathbb{E} U^{-\nu-1} e^{-p/U} = \int_0^\infty \left( \frac{s}{p} \right)^{\frac{\nu}{2}} I_\nu(2\sqrt{ps}) \mathbb{E} U^s e^{-sU} ds, \quad \nu > -1. \tag{5.3}
\]

(Oberhettinger & Badii do not specify for which \( p \) this holds, but this is clarified in the theorem below.) Observe that \( \nu = -1 \) is not allowed here, so that (5.3) does not apply to \( \mathbb{E} e^{-p/U} \). We now derive some more general formulas, which have (5.1) and (5.3) as particular cases.

**Theorem 5.1.** Suppose \( p > 0 \), \( r \in \mathbb{R} \) and \( U > 0 \) a.s.

(a) If \( \nu > -1 \), then

\[
\mathbb{E} U^{r-\nu-1} e^{p/U} = \int_0^\infty \left( \frac{s}{p} \right)^{\frac{r}{2}} I_\nu(2\sqrt{ps}) \mathbb{E} U^r e^{-sU} ds.
\]

Both sides may be infinite.

(b) If \( \nu \leq -1 \), then

\[
\mathbb{E} U^{r-\nu-1} e^{p/U} = \sum_{0 \leq n \leq -\nu-1} \frac{p^n}{n!} \mathbb{E} U^{r-\nu-n-1} + \int_0^\infty \left( \frac{s}{p} \right)^{\frac{r}{2}} I_\nu(2\sqrt{ps}) - \sum_{0 \leq n \leq -\nu-1} \frac{p^n s^{n+\nu}}{n!\Gamma(n+\nu+1)} \right] \mathbb{E} U^r e^{-sU} ds.
\]

Both sides may be infinite. If, moreover, \( -\nu-1 \in \mathbb{N} \), then

\[
\mathbb{E} U^{r-\nu-1} e^{p/U} = \sum_{0 \leq n \leq -\nu-1} \frac{p^n}{n!} \mathbb{E} U^{r-\nu-n-1} + \int_0^\infty \left( \frac{s}{p} \right)^{\frac{r}{2}} I_{-\nu}(2\sqrt{ps}) \mathbb{E} U^r e^{-sU} ds.
\]

(c) If \( \nu > -1 \), \( \mathbb{E} U^r < \infty \), \( \mathbb{E} U^{r-\nu-1} < \infty \), then

\[
\mathbb{E} U^{r-\nu-1} e^{p/U} = \int_0^\infty \left( \frac{s}{p} \right)^{\frac{r}{2}} J_\nu(2\sqrt{ps}) \mathbb{E} U^r e^{-sU} ds.
\]

Both sides are finite, and the integral on the right may be improper (that is, it may not converge absolutely) if \( \nu < -1/2 \).
The integrated square-root process

(d) If \( \nu \leq -1 \), \( \mathbb{E} U^r < \infty \), \( \mathbb{E} U^{r-\nu-1} < \infty \), then

\[
\mathbb{E} U^{r-\nu-1} e^{-p/U} = \sum_{0 \leq n \leq -\nu-1} \frac{(-p)^n}{n!} \mathbb{E} U^{r-\nu-n-1}
\]

\[
+ \int_0^\infty \left[ \left( \frac{s}{p} \right)^\nu J_\nu(2\sqrt{ps}) - \sum_{0 \leq n \leq -\nu-1} \frac{(-p)^n s^{n+\nu}}{n! \Gamma(n+\nu+1)} \right] \mathbb{E} U^r e^{-sU} \, ds.
\]

Both sides are finite, and the integral on the right may be improper. If, moreover, \( -\nu - 1 \in \mathbb{N} \), then

\[
\mathbb{E} U^{r-\nu-1} e^{-p/U} = \sum_{0 \leq n \leq -\nu-1} \frac{(-p)^n}{n!} \mathbb{E} U^{r-\nu-n-1} + (-1)^\nu \int_0^\infty \left( \frac{s}{p} \right)^\nu J_\nu(2\sqrt{ps}) \mathbb{E} U^r e^{-sU} \, ds.
\]

**Proof.** (a) Because all the terms in the following series are positive, the equalities hold whether the resulting expressions are finite or infinite:

\[
\mathbb{E} U^{r-\nu-1} e^{p/U} = \sum_{n=0}^\infty \frac{p^n}{n!} \mathbb{E} U^{r-\nu-n-1}
\]

\[
= \sum_{n=0}^\infty \frac{p^n}{n!} \mathbb{E} U^r \int_0^\infty \frac{s^{n+\nu}}{\Gamma(n+\nu+1)} e^{-sU} \, ds
\]

\[
= \int_0^\infty \left( \sum_{n=0}^\infty \frac{p^n}{n!} \frac{s^{n+\nu}}{\Gamma(n+\nu+1)} \right) \mathbb{E} U^r e^{-sU} \, ds
\]

\[
= \int_0^\infty \left( \frac{s}{p} \right)^\nu I_\nu(2\sqrt{ps}) \mathbb{E} U^r e^{-sU} \, ds.
\]

(b) The first formula results from

\[
\mathbb{E} U^{r-\nu-1} e^{p/U} = \sum_{0 \leq n \leq -\nu-1} \frac{p^n}{n!} \mathbb{E} U^{r-\nu-n-1} + \int_0^\infty \sum_{n>-\nu-1} \frac{p^n s^{n+\nu}}{n! \Gamma(n+\nu+1)} \mathbb{E} U^r e^{-sU} \, ds.
\]

The second one then follows from (Lebedev, 1972, p.110)

\[
\frac{1}{\Gamma(-m)} = 0, \quad I_{-m}(z) = I_m(z), \quad m \in \mathbb{N}.
\]

(c) Let \( U_\epsilon = \max(U, \epsilon) \) for \( \epsilon \geq 0 \) (whence \( U_0 = U \)). For \( \epsilon > 0 \), \( 1/U_\epsilon \) is bounded below by 0, and above by \( 1/\epsilon \), whence

\[
\mathbb{E} U^{r}_{\epsilon} U_{\epsilon}^{r-\nu-1} e^{p/U_\epsilon} < \infty.
\]
The integrated square-root process

Therefore the series below converge absolutely, and so it is permitted to reverse the order of summation, integration and expectation at will:

\[
E U_r^{-\nu - 1} e^{-p/U} = \int_0^\infty \left( \sum_{n=0}^\infty \frac{(-p)^n}{n!} \frac{s^{n+\nu}}{\Gamma(\nu + n + 1)} \right) E U_r^{-sU} ds \\
= \int_0^\infty \left( \frac{s^\nu}{p} \right) J_\nu(2\sqrt{ps}) E U_r^{-sU} ds. \tag{5.4}
\]

Now \(-\nu - 1 < 0\), so

\[
0 \leq U_r^{-\nu - 1} e^{-p/U} \leq C U_r
\]

where \(C = \sup_{x > 0} x^{\nu + 1} e^{-px} < \infty\), and \(U_r\) is integrable by assumption. Thus

\[
E U_r^{-\nu - 1} e^{-p/U} \to E U_r e^{-p/U}
\]
as \(\epsilon \to 0^+\). Next, turn to the right hand side of (5.4).

**Case 1: \(\nu \geq -\frac{1}{2}\)**

The asymptotic expression (Lebedev, 1972, p.122)

\[
J_\nu(x) = \left( \frac{2}{\pi x} \right)^{\frac{1}{2}} \cos \left( x - (2\nu + 1)\frac{\pi}{4} \right) + O(x^{-3/2}) \quad \text{as } x \to \infty
\]

implies that there is \(C > 0\) such that

\[
|J_\nu(x)| \leq Cx^{-\frac{1}{2}}, \quad x > 0.
\]

Therefore

\[
\left| \left( \frac{s}{p} \right)^\nu J_\nu(2\sqrt{ps}) E U_r^{-sU} \right| \leq C_1 s^{\frac{\nu}{2} - \frac{3}{4}} E U_r^{-sU}.
\]

Now \(\nu \geq -\frac{1}{2}\) implies \(\nu + \frac{3}{4} > 0\), and so

\[
\int_0^\infty s^{\frac{\nu}{2} - \frac{3}{4}} E U_r^{-sU} ds = E U_r^{-\frac{\nu}{2} - \frac{3}{4}} \Gamma \left( \frac{\nu}{2} + \frac{3}{4} \right).
\]

This is finite, because \(r - \nu - 1 \leq r - \frac{\nu}{2} - \frac{3}{4} \leq r\), and it is assumed that both \(E U_r\) and \(E U_r^{-\nu - 1}\) are finite. Finally, the right hand side of (5.4) converges to

\[
\int_0^\infty \left( \frac{s}{p} \right)^\nu J_\nu(2\sqrt{ps}) E U_r^{-sU} ds
\]
as \(\epsilon \to 0^+\), by dominated convergence.
The integrated square-root process

Case 2: $\nu \in (-1, -\frac{1}{2})$ The above arguments break down if $\nu < -\frac{1}{2}$, because then

$$r - \frac{\nu}{2} - \frac{3}{4} < r - \nu - 1.$$  

The problem can be solved either by assuming that $\mathbb{E}U^{r - \frac{\nu}{2} - \frac{3}{4}} < \infty$ or else as follows.

Write the integral on the right hand side of (5.4) as $\int_0^1 + \int_1^\infty$. It is immediately seen that

$$\int_0^1 \left( \frac{s}{p} \right)^{\frac{\nu}{2}} J_{\nu}(2\sqrt{ps}) \mathbb{E}U^r e^{-sU^e} ds \to \int_0^1 \left( \frac{s}{p} \right)^{\frac{\nu}{2}} J_{\nu}(2\sqrt{ps}) \mathbb{E}U^r e^{-sU} ds$$

as $\epsilon \to 0+$, by dominated convergence. For the second integral, consider separately each term on the right of

$$J_{\nu}(2\sqrt{ps}) = C_2 s^{-\frac{1}{4}} \cos \left( 2\sqrt{ps} - (2\nu + 1) \frac{\pi}{4} \right) + R(s) \quad \text{as } s \to \infty$$

where $C_2 > 0$ and $R(s) = O(s^{-\frac{1}{4}}).$ Since $\frac{\nu}{2} - \frac{3}{4} < -1$, the integral

$$\int_1^\infty \left( \frac{s}{p} \right)^{\frac{\nu}{2}} R(s) \mathbb{E}U^r e^{-sU^e} ds$$

converges absolutely for $\epsilon \geq 0$, and so convergence as $\epsilon \to 0+$ to

$$\int_1^\infty \left( \frac{s}{p} \right)^{\frac{\nu}{2}} R(s) \mathbb{E}U^r e^{-sU} ds$$

follows at once. The only remaining problem is to show that

$$\int_0^\infty s^{\frac{\nu}{2} - \frac{1}{4}} \cos \left( 2\sqrt{ps} - (2\nu + 1) \frac{\pi}{4} \right) \mathbb{E}U^r e^{-sU^e} ds$$

converges to the same expression with $\epsilon = 0$, as $\epsilon \to 0+$. Perform the change of variable $u = 2\sqrt{ps}$, $s = u^2/(4p)$, to obtain

$$2 \int_{2\sqrt{p}}^\infty \left( \frac{u^2}{4p} \right)^{\frac{\nu}{2} - \frac{1}{4}} u \cos(u - (2\nu + 1) \frac{\pi}{4}) \mathbb{E}U^r e^{-u^2U^e/(4p)} du$$

$$= C_3 \int_{2\sqrt{p}}^\infty u^{\nu + \frac{1}{2}} \cos(u - (2\nu + 1) \frac{\pi}{4}) \mathbb{E}U^r e^{-u^2U^e/(4p)} du$$

$$= C_3 \left\{ u^{\nu + \frac{1}{2}} \sin(u - (2\nu + 1) \frac{\pi}{4}) \mathbb{E}U^r e^{-u^2U^e/(4p)} \bigg|_{u=2\sqrt{p}}^{u \to \infty} \right\}$$

$$- C_3 \int_{2\sqrt{p}}^\infty \sin(u - (2\nu + 1) \frac{\pi}{4}) d \left[ u^{\nu + \frac{1}{2}} \mathbb{E}U^r e^{-u^2U^e/(4p)} \right].$$

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Now \( u^{\nu + \frac{1}{2}} \) tends to 0 as \( u \) tends to \( \infty \), so the expression inside the curly brackets reduces to

\[
- u^{\nu + \frac{1}{2}} \sin(u - (2\nu + 1) \frac{\pi}{4}) E^r e^{-u^2 U_e/(4p)} \bigg|_{u=2\sqrt{p}},
\]

which poses no problem. The integral in (5.5) converges absolutely, since

\[
\lim_{u \to \infty} u^{\nu + \frac{1}{2}} E^r e^{-u^2 U_e/(4p)} = 0.
\]

To prove that it converges to

\[
\int_{2\sqrt{p}}^{\infty} \sin(u - (2\nu + 1) \frac{\pi}{4}) d \left[ u^{\nu + \frac{1}{2}} E^r e^{-u^2 U_e/(4p)} \right],
\]

apply the Helly-Bray Theorem (Loève, 1977, p.184; Kolmogorov & Fomine, 1975, p.370). The functions

\[
F_\epsilon(u) = u^{\nu + \frac{1}{2}} E^r e^{-u^2 U_e/(4p)}, \quad \epsilon \geq 0
\]

are of bounded variation on \( \mathbb{R}_+ \), and \( \{F_\epsilon; \epsilon > 0\} \) converge completely (Loève, 1977, p.180) to \( F_0 \) as \( \epsilon \to 0^+ \). Since \( g(u) = \sin(u - (2\nu + 1) \frac{\pi}{4}) \) is bounded, we conclude that

\[
\int_{2\sqrt{p}}^{\infty} g \, dF_\epsilon \to \int_{2\sqrt{p}}^{\infty} g \, dF_0 \quad \text{as} \quad \epsilon \to 0^+.
\]

(d) Proceed as in (b) and (c) to get

\[
E^r U_e^{-\nu - 1} e^{p/U_e} = \sum_{0 \leq n \leq -\nu - 1} \frac{(-p)^n}{n!} E^r U_e^{-\nu - n - 1}
\]

\[
+ \int_0^{\infty} \left[ \left( \frac{s}{p} \right)^{\frac{\nu}{2}} J_{\nu}(2\sqrt{ps}) - \sum_{0 \leq n \leq -\nu - 1} \frac{(-p)^n s^{\nu + n}}{n! \Gamma(\nu + n + 1)} \right] E^r e^{-s U_e} \, ds
\]

for any \( \epsilon > 0 \). The first sum poses no problem. Split the integral into \( \int_0^1 + \int_1^{\infty} \). The former tends to the right limit by dominated convergence. To prove convergence of \( \int_1^{\infty} \), consider each term of the integrand separately. The first one converges to

\[
\int_1^{\infty} \left( \frac{s}{p} \right)^{\frac{\nu}{2}} J_{\nu}(2\sqrt{ps}) E^r e^{-s U_e} \, ds
\]

by the arguments given in Case 2 of (c). Each of the other terms gives rise to an absolutely convergent integral, since \( n + \nu < -1 \), and dominated convergence does the rest.

Finally, in the special case \( -\nu - 1 \in \mathbb{N} \), the sum inside the integral vanishes (as in (b)), and one applies (Lebedev, 1972, p.103)

\[
J_{-n}(z) = (-1)^n J_n(z), \quad n = 1, 2, \ldots
\]

\( \square \)
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Theorem 4.1(b) results from letting $\nu = -1$ in Theorem 5.1(b). This theorem has the following consequences, which relate the existence of $\mathbb{E} U^{-\nu - 1} e^{p/U}$, which is itself determined by the behaviour of the distribution of $U$ near the origin, to the behaviour of $\mathbb{E} e^{-sU}$ near infinity.

**Corollary 5.2.** Suppose $p > 0$, $\nu \in \mathbb{R}$ and $U > 0$ a.s.

(a) If $\mathbb{E} U^{-\nu - 1} e^{p/U} < \infty$, then

$$\liminf_{s \to \infty} s^{\frac{\nu}{2} - \frac{1}{4}} e^{2\sqrt{ps}} \mathbb{E} e^{-sU} = 0.$$  

(b) If $\nu > -1$ and if, for some $\epsilon > 0$, $\mathbb{E} e^{-sU} = \mathcal{O}(s^{-\frac{3}{4} - \frac{\nu}{2} - \epsilon} e^{-2\sqrt{ps}})$ as $s \to \infty$, then $\mathbb{E} U^{-\nu - 1} e^{p/U} < \infty$. The same holds for $\nu \leq -1$, with the additional assumption $\mathbb{E} U^{-\nu - 1} < \infty$.

**Proof.** (a) First suppose $\nu > -1$, and let $r = 0$ in Theorem 5.1(a). If the integral converges, then necessarily the integrand must have a limit inferior equal to 0. By (4.3),

$$\liminf_{s \to \infty} s^{\frac{\nu}{2}} I_{\nu}(2\sqrt{ps}) \mathbb{E} e^{-sU} = C \liminf_{s \to \infty} s^{\frac{\nu}{2} - \frac{1}{4}} e^{2\sqrt{ps}} \mathbb{E} e^{-sU},$$

where $C > 0$ is a constant. For $\nu \leq -1$, let $r = 0$ in Theorem 5.1(b). If the left hand side of the first formula is finite, then the integral on the other side must be finite. The situation is seen to be the same as when $\nu > -1$, once it is observed that

$$\int_{1}^{\infty} s^{n+\nu} \mathbb{E} e^{-sU} ds < \infty$$

for any $n < \nu + 1$.

(b) Suppose $\nu > -1$. The assumptions imply that there is a constant $C$ such that

$$\mathbb{E} U^{-\nu - 1} e^{p/U} = \left[ \int_{0}^{1} + \int_{1}^{\infty} \right] \left( \frac{s}{p} \right)^{\nu} I_{\nu}(2\sqrt{ps}) \mathbb{E} U^{r} e^{-sU} ds$$

$$\leq \int_{0}^{1} \left( \frac{s}{p} \right)^{\nu} I_{\nu}(2\sqrt{ps}) \mathbb{E} U^{r} e^{-sU} ds + C \int_{1}^{\infty} s^{\frac{\nu}{2}} I_{\nu}(2\sqrt{ps}) s^{-\frac{3}{4} - \frac{\nu}{2} - \epsilon} e^{-2\sqrt{ps}} ds$$

$$= \int_{0}^{1} \left( \frac{s}{p} \right)^{\nu} I_{\nu}(2\sqrt{ps}) \mathbb{E} U^{r} e^{-sU} ds + C \int_{1}^{\infty} s^{\frac{1}{4} - 2\sqrt{ps}} I_{\nu}(2\sqrt{ps}) s^{-1-\epsilon} ds$$

The result follows from (4.3). If $\nu \leq -1$ and, moreover, $\mathbb{E} U^{-\nu - 1} < \infty$, then

$$\mathbb{E} U^{-\nu - n - 1} < \infty \quad \forall \ 0 \leq n \leq -\nu - 1,$$

$$\int_{1}^{\infty} s^{n+\nu} \mathbb{E} e^{-sU} ds < \infty \quad \forall \ 0 \leq n < -\nu - 1,$$
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and

\[ \int_1^\infty s^{\frac{\nu}{2}} I_\nu(2\sqrt{ps}) \mathbb{E} e^{-sU} \, ds < \infty. \]

This implies \( \mathbb{E} U^{-\nu-1} e^{p/U} < \infty \), by Theorem 5.1(b).

**Example.** Macdonald’s function is defined in terms of the modified Bessel function as follows (Lebedev, 1972):

\[ K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu \pi)}, \quad |\arg z| < \pi, \quad \nu \notin \mathbb{Z}, \]

\[ K_n(z) = \lim_{\nu \to n} K_\nu(z), \quad n \in \mathbb{Z}. \]

It also has the integral expression

\[ K_\nu(z) = \frac{1}{2} \left( \frac{z}{2} \right)^\nu \int_0^\infty e^{-t-(z^2/4t)} t^{-\nu-1} \, dt, \quad |\arg(z)| < \frac{\pi}{4}. \]

Clearly \( K_{-\nu}(z) = K_\nu(z) \). The generalized inverse Gaussian distribution with parameters \( \alpha > 0, \beta > 0, \gamma \in \mathbb{R} \) has density

\[ f(x) = \left( \frac{\alpha}{\beta} \right)^{\frac{\gamma}{2}} \frac{1}{2K_\gamma(\sqrt{\alpha \beta})} x^{\gamma-1} \exp \left\{ -\frac{1}{2} \left( \frac{\alpha x + \beta}{x} \right) \right\} 1_{(0,\infty)}(x). \]

The notation for this distribution is \( \text{GIG}(\alpha, \beta, \gamma) \). It is immediate that \( U \sim \text{GIG}(\alpha, \beta, \gamma) \) implies \( (1/U) \sim \text{GIG}(\beta, \alpha, -\gamma) \). Moreover, a simple calculation shows that

\[ \mathbb{E} U^r e^{-sU} = \frac{\alpha^{\frac{r}{2}} \beta^{\frac{r}{2}}}{(\alpha + 2s)^{\frac{r+\gamma}{2}}} \frac{K_{\gamma+r}(\sqrt{(\alpha + 2s)\beta})}{K_\gamma(\sqrt{\alpha \beta})}, \quad \Re(s) > -\frac{\alpha}{2}. \]

Theorem 5.1(c) then says that

\[ \int_0^\infty \left( \frac{s}{p} \right)^{\frac{\gamma}{2}} J_\nu(2\sqrt{ps}) \frac{\alpha^{\frac{\gamma}{2}} \beta^{\frac{r}{2}}}{(\alpha + 2s)^{\frac{r+\gamma}{2}}} \frac{K_{\gamma+r}(\sqrt{(\alpha + 2s)\beta})}{K_\gamma(\sqrt{\alpha \beta})} \, ds \]

\[ = \frac{\beta^{\frac{\gamma}{2}} \alpha^{\frac{1+\nu-r}{2}}}{(\beta + 2p)^{-\gamma+\frac{\nu-r}{2}}} \frac{K_{1+\nu-r-\gamma}(\sqrt{(\beta + 2p)\alpha})}{K_\gamma(\sqrt{\alpha \beta})} \]

or, equivalently,

\[ \int_0^\infty \left( \frac{s}{p} \right)^{\frac{\gamma}{2}} J_\nu(2\sqrt{ps}) \frac{K_{\gamma+r}(\sqrt{(\alpha + 2s)\beta})}{(\alpha + 2s)^{\frac{r+\gamma}{2}}} \, ds \]

\[ = \frac{\beta^{\frac{\gamma+r}{2}} \alpha^{1+\nu-r-\gamma}}{(\beta + 2p)^{-\gamma+1+\nu-r}} K_{1+\nu-r-\gamma}(\sqrt{(\beta + 2p)\alpha}). \quad (5.6) \]
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By performing the change of variable \( x = 2\sqrt{p\bar{z}}/b \), and setting

\[
\alpha = \frac{b^2 y^2}{2p}, \quad \beta = \frac{2a^2 p}{b^2}, \quad \mu = \gamma + r,
\]

it can be seen that (5.6) is equivalent to the formula (Lebedev, 1972, p. 134)

\[
\int_0^\infty K_{\mu}(a\sqrt{x^2 + y^2}) J_\nu(bx) x^{\nu+1} \, dx = \frac{b^\nu}{a^\mu} \left( \frac{\sqrt{a^2 + b^2}}{y} \right)^{\mu-\nu-1} K_{\mu-\nu-1}(y\sqrt{a^2 + b^2}). (5.7)
\]

An extension of this result is obtained by applying part (d) of Theorem 5.1 to the same distribution \( \text{GIG}(\alpha, \beta, \gamma) \). With the same change of variable and substitutions, we get

\[
\int_0^\infty K_{\mu}(a\sqrt{x^2 + y^2}) \left( J_\nu(bx) - \sum_{0 \leq n < -\nu-1} \frac{(-1)^n (bx/2)^{\nu+2n}}{n! \Gamma(n+\nu+1)} \right) x^{\nu+1} \, dx
\]

\[
= \frac{b^\nu}{a^\mu} \left( \frac{\sqrt{a^2 + b^2}}{y} \right)^{\mu-\nu-1} K_{\mu-\nu-1}(y\sqrt{a^2 + b^2}) - \frac{b^\nu}{a^{\nu+1}y^{\mu-\nu-1}} \sum_{0 \leq n \leq -\nu-1} \frac{1}{n!} \left( -\frac{b^2 y}{2a} \right)^n K_{\mu-\nu-n-1}(ay). (5.8)
\]

The Appendix shows another way of deriving the same formula.

6. Conclusion

This paper has shown how the moments of the integral of the square-root process may be computed. Of the three methods shown, the the recursive method (Theorem 3.2) is the fastest to execute, and its programming poses no problem. The paper also showed some regularly properties of the density of the integral.

Further work will consider expressing the density of \( Y_t \) as an infinite Laguerre series, as was done for the density of the integral of geometric Brownian motion in Dufresne (2000). The coefficients of the Laguerre polynomials are combinations of moments of the variable, and this is where the recursive equations of Section 3 will be essential, as they generate the moments much faster than differentiating the Laplace transform. A finite number of terms of the series would then serve as an approximation for the density. The same approach may be used for options on integrated squared volatility, or for options on average interest rates. Alternative methods are inversion of the Laplace transform, and Monte Carlo simulation.

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References


APPENDIX

We give another proof of formula (5.8). First, we derive a slight extension of Weber’s integral (Lebedev, 1972, p.132)

$$\int_0^\infty e^{-a^2x^2} J_\nu(bx)x^{\nu+1} dx = \frac{b^\nu}{(2a^2)^{\nu+1}} e^{-b^2/4a^2}, \quad a, b > 0, \quad \text{Re}(\nu) > -1.$$  

Leave $a, b > 0$, but now suppose $\nu \leq -1$ (the case of complex $\nu$ with Re($\nu$) $\leq -1$ remains to be explored). Consider

$$\int_0^\infty e^{-a^2x^2} \left[ J_\nu(bx) - \sum_{0 \leq n \leq -\nu-1} \frac{(-1)^n (bx/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)} \right] x^{\nu+1} dx$$

$$= \sum_{n > -\nu-1} \frac{(-1)^n (b/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)} \int_0^\infty e^{-a^2x^2} x^{\nu+2n+1} dx$$

$$= \sum_{n > -\nu-1} \frac{(-1)^n (b/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)} \frac{1}{2} a^{-2\nu-2n-2} \int_0^\infty e^{-u^{\nu+n}} du$$

$$= \frac{b^\nu}{(2a^2)^{\nu+1}} \left[ e^{-b^2/4a^2} - \sum_{0 \leq n \leq -\nu-1} \frac{1}{n!} \left( -\frac{b^2}{4a^2} \right)^n \right].$$

From this formula, we obtain

$$\int_0^\infty K_\mu\left(\frac{a\sqrt{x^2 + y^2}}{x^2 + y^2}\right) \left[ J_\nu(bx) - \sum_{0 \leq n \leq -\nu-1} \frac{(-1)^n (bx/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)} \right] x^{\nu+1} dx$$

$$= \int_0^\infty dx \left[ J_\nu(bx) - \sum_{0 \leq n \leq -\nu-1} \frac{(-1)^n (bx/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)} \right] x^{\nu+1} \frac{a^\mu}{2^\mu+1} \int_0^\infty \frac{dt}{t^{\mu+1}} e^{-t-a^2(x^2+y^2)/4t}$$

$$= \frac{a^\mu}{2^\mu+1} \int_0^\infty \frac{dt}{t^{\mu+1}} e^{-t-a^2y^2/4t} \int_0^\infty dx e^{-a^2x^2/4t} \left[ J_\nu(bx) - \sum_{0 \leq n \leq -\nu-1} \frac{(-1)^n (bx/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)} \right] x^{\nu+1}$$

$$= \frac{a^\mu}{2^\mu+1} \int_0^\infty \frac{dt}{t^{\mu+1}} e^{-t-a^2y^2/4t} \frac{b^\nu}{(a^2/2t)^{\nu+1}} \left[ e^{-b^2t/a^2} - \sum_{0 \leq n \leq -\nu-1} \frac{1}{n!} \left( -\frac{b^2t}{a^2} \right)^n \right]$$

$$= 2^{\nu-\mu} a^{\mu-2\nu-2} b^\nu \left[ \int_0^\infty \frac{dt}{t^{\mu-\nu}} e^{-t-a^2y^2/4t-b^2t/a^2} - \sum_{0 \leq n \leq -\nu-1} \frac{1}{n!} \left( -\frac{b^2y}{2a} \right)^n \int_0^\infty \frac{dt}{t^{\mu-\nu-1}} e^{-t-a^2y^2/4t} \right]$$

$$= \frac{b^\nu}{a^\mu} \left( \frac{\sqrt{a^2 + b^2}}{y} \right)^{\nu-\mu-1/2} K_{\mu-\nu-1}(y\sqrt{a^2 + b^2}) - \frac{b^\nu}{a^{\mu+1} y^{\nu-1}} \sum_{0 \leq n \leq -\nu-1} \frac{1}{n!} \left( -\frac{b^2y}{2a} \right)^n K_{\mu-\nu-1-1}(ay).$$