THE INTEGRATED SQUARE-ROOT PROCESS

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Abstract

This paper studies some of the properties of the square-root process, particularly those of the integral of the process over time. After summarizing the properties of the square-root process, the Laplace transform of the integral of the square-root process is derived. Three methods for the computation of the moments of this integral are given, as well as some properties of the density of the integral. The relationship between the Laplace transforms of a variable and of its reciprocal is studied. An application to the generalized inverse Gaussian distribution is given.

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1. Introduction

The *square-root process* is the unique strong solution of the stochastic differential equation (SDE in the sequel)

$$dX \;=\; (\alpha X+\beta)\,dt+\gamma\sqrt{X}\,dW, \qquad X_0=\bar{x}\geq 0, \tag{1.1}$$

where $\alpha\in\mathbb{R},\beta>0,\gamma>0,\bar{x}\geq 0$, and W is standard Brownian motion. Other names used for the same process are *squared Gaussian* (since the square of some Gaussian processes can be shown to satisfy (1.1) — a comment attributed to L.C.G. Rogers), *Feller* (because of the early work of William Feller on the process), *CIR* (since Cox, Ingersoll & Ross (1985) used (1.1) as a model for the spot interest rate, and gave the formulas for risk-free bonds under this assumption), and *Heston model* (since Heston (1993) studied the pricing of options when the squared volatility of the underlying security satisfies (1.1)).

The integral of X arises in both volatility and interest rate models. If the stock price follows the process

$$dS \;=\; \mu S\,dt+\sqrt{X}S\,dV,$$

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where (V, W) is two-dimensional Brownian motion (possibly correlated), and X is the solution of (1.1), then the integral

$$Y_t \;=\; \int_0^t X_s\, ds$$

is the cumulative variance of the stock returns up to time t. This integrated variance process occurs in volatility swaps (Carr, 2001), and also in other derivatives (see below). The reason for this research is the quest for explicit formulas for options with payoffs involving Y_t. The results in this paper have already been used by the author for this purpose; this will be the subject of a subsequent paper.

The process Y_t obviously arises in bond pricing, when spot interest rates follow the CIR model. This is because the price of a risk-free zero-coupon bond maturing at time t equals

$$\mathsf{E}_{\mathsf{Q}}\, e^{-Y_t}$$

where Q is a risk-neutral measure. Another example would be options based on the average spot interest rate, for instance a call on the average of spot rates X_t over $[0, T]$, with strike R_0:

$$e^{-\int_0^T X_t\, dt}\left(\frac{1}{T}\int_0^T X_t\, dt - R_0\right)_+ \;=\; e^{-Y_t}\frac{1}{T}\left(Y_T - TR_0\right)_+$$

The price of such an option would be the expectation of this discounted payoff with respect to the risk neutral measure, which requires knowledge of the distribution of Y_T.

The moments and distribution of Y_t may also be of interest in the statistical estimation of the parameters α, β, γ in (1.1).

Here is a summary of the paper. Section 2 shows that the moments of X_t are solutions of a sequence of ordinary differential equations, and that these moments allow the determination of the moment generating function and of the density of the process. The moments of X_t are required in the sequel, and the technique used to obtain them is used again in Section 3. Moreover, Section 2 yields an improvement of a result due to Dufresne (1989), regarding the moments of a process related to Asian options and the Courtadon interest rate model. The density of X_t in (1.1) is of course well known, but the derivation given in Section 2 is not common (the author has not seen it in the literature), and clearly shows that the distribution of the process is the convolution of two distributions, one compound Poisson/Exponential and the other gamma. (The distribution of X_t has been called "non-central chi-square," apparently because it reduces to the distribution of the square of non-zero normal variable when $\alpha = 0$.) The better known way of finding the moment generating function and density of X_t is via the relationship with Bessel processes, and this is also indicated in Section 2.

Section 3 shows how the moments of the integral of the square-root process may be obtained, either by solving linear differential equations, or more directly by a recursive procedure; the results are apparently new. Section 4 is a derivation of the Laplace transform of the integral of the square-root process, based on the formula for the price of a zero-coupon bond in the CIR framework (the latter was based on results previously known in probability theory). The Laplace transform yields a third way of calculating the moments. Section 5 extends a result which is used in Section 4, and is of independent interest in Probability Theory. Formulas are given which relate the Laplace transform of a random variable $U > 0$ to the Laplace transform of $1/U$. They extend one well-known formula for Laplace transforms. An application to the generalized inverse Gaussian distribution is given.

2. A sequence of ODEs; the moments and distribution of X_t

When the square-root process (1.1) is used as a financial model, it is usually made "mean-reverting" by letting $\alpha < 0$. In the sequel no such restriction is imposed, except that Theorem 2.3 assumes $\alpha \neq 0$.

The moments of X_t are easy to obtain recursively, by applying Ito's formula and then taking epectations. A general formula for the moments of X_t will now be derived. The technique is similar to the one used in Dufresne (1989, 1990) for the integral of geometric Brownian motion, and yields an extension of one result in Dufresne (1989) (Corollary 2.2 below). Next, the Laplace transform of the density of X_t is derived, based on the moments only. This derivation is different form the usual ones, which are either based on PDEs, or on the relationship with Bessel processes. Our derivation is based on the assumption that all the moments of X are finite. The moments are calculated and then summed to get the moment generating function (MGF). Finally, the density is exhibited, and is shown to be the convolution of a Gamma and a compound Poisson/exponential.

(*N.B.* The MGF of a probability measure m is the mapping

$$r \mapsto \int e^{rx} m(dx).$$

The concept is thus the same as the Laplace transform, but a $+$ instead of a $-$ in front of the argument.)

Applying Ito's formula with $f(x) = x^k$ to (1.1), we get

$$dX^k = (a_k X^k + b_k X^{k-1})dt + \gamma k X^{k-\frac{1}{2}} dW_t \tag{2.1}$$

with

$$a_k = \alpha k, \qquad b_k = \beta k + \frac{1}{2}\gamma^2 k(k-1).$$

Eq.(2.1) is the same as

$$X_t^k = \bar{x}^k + \int_0^t (a_k X_s^k + b_k X_s^{k-1}) ds + \int_0^t \gamma k X_s^{k-\frac{1}{2}} dW_s.$$

Here we let the initial value of the process be $\bar{x} \geq 0$, and we take as given that $\mathsf{E} X^n < \infty$ for all $n \geq 0$. The consequence is that the Ito integral above has expected value 0. Defining

$m_k(t) = \mathsf{E}X_t^k$, $k = 0, 1, \ldots$, we can then differentiate with respect to t on each side to obtain

$$\begin{aligned} m_k'(t) &= a_k m_k(t) + b_k m_{k-1}(t), \quad k \geq 1,\ t > 0 && (2.2) \\ m_k(0) &= \bar{x}^k, \quad k \geq 0. && (2.3) \end{aligned}$$

Eq. (2.2) is of the same type as Eq.(18) in Dufresne (1989), and can be solved in the same way. The main difference here is that the initial value of $m_k(t)$ is not 0.

Theorem 2.1. *Let $a_0 = 0$ and suppose the numbers $\{a_0, \ldots, a_K\}$ are distinct. Then the solution of (2.2), subject to (2.3) and $m_0(t) \equiv 1$ is*

$$m_k(t) = \sum_{j=0}^{k} d_{kj} e^{a_j t}, \qquad k = 0, \ldots, K, \tag{2.4}$$

where

$$d_{kj} = \sum_{i=0}^{j} \bar{x}^i \left(\prod_{m=i+1}^{k} b_m \right) \prod_{\substack{\ell=i \\ \ell \neq j}}^{k} \frac{1}{a_j - a_\ell}, \qquad j = 0, \ldots, k. \tag{2.5}$$

Proof. Define a product over an empty set of indices as 1. Then formulas (2.4) and (2.5) are correct for $k = 0$. For $k = 1$, (2.2) implies

$$m_1(t) = m_1(0)e^{a_1 t} + b_1 e^{a_1 t} \int_0^t e^{-a_1 s} ds = \bar{x}e^{a_1 t} + b_1 \frac{e^{a_1 t} - 1}{a_1} = -\frac{b_1}{a_1} + \left(\frac{b_1}{a_1} + \bar{x} \right) e^{a_1 t}.$$

For $j = 0$ and $j = 1$, $m_j(t)$ is a combination of $e^{a_\ell t}$, $\ell \leq j$. If this claim is correct for $j = 0, \ldots, k-1$, then $m_k(t)$ is the solution of an inhomogeneous ordinary differential equation with a forcing term equal to a combination of exponentials:

$$\begin{aligned} m_k'(t) - a_k m_k(t) &= \sum_{j=0}^{k-1} C_j e^{a_j t} \\ \Longrightarrow \qquad m_k(t) &= m_k(0)e^{a_k t} + e^{a_k t} \int_0^t e^{-a_k s} \sum_{j=0}^{k-1} C_j e^{a_j s} ds. && (2.6) \end{aligned}$$

Since the $\{a_0, \ldots, a_K\}$ are distinct, $m_k(t)$ has to be of the form

$$d_{k0} + d_{k1}e^{a_1 t} + \cdots + d_{kk}e^{a_k t}.$$

By induction, this is true for $k = 0, \ldots, K$. As a function of $\bar{x}$, $m_k(t)$ is a polynomial of degree k, and the coefficient of $\bar{x}^k$ in $m_k(t)$ is $e^{a_k t}$ (because $\bar{x}^k$ is the initial condition $m_k(0)$).

Insert (2.4) into (2.2) to obtain

$$\sum_{j=0}^{k} a_j d_{kj} e^{a_j t} \;=\; \sum_{j=0}^{k} a_k d_{kj}\, e^{a_j t} + \sum_{j=0}^{k-1} b_k d_{k-1,j}\, e^{a_j t},$$

which implies

$$d_{kj} \;=\; \frac{b_k}{a_j - a_k} d_{k-1,j} \;=\; \frac{b_k}{a_j - a_k} \cdots \frac{b_{j+1}}{a_j - a_{j+1}} d_{jj}, \qquad 0 \le j < k. \tag{2.7}$$

Since $m_j(t)$ is a polynomial of degree j in $\bar{x}$, the last equation implies that d_{kj} is also a polynomial of degree j in $\bar{x}$, which we write as

$$d_{kj} \;=\; \sum_{i=0}^{j} d_{kji}\, \bar{x}^i.$$

By identifying the coefficients of $\bar{x}^i$ in (2.7), we get

$$d_{kji} \;=\; \frac{b_k}{a_j - a_k} \cdots \frac{b_{j+1}}{a_j - a_{j+1}} d_{jji}, \qquad 0 \le i \le j < k.$$

We also know that $d_{kkk} = 1$, since the leading coefficient of $m_k(t)$, as a polynomial in $\bar{x}$, is $e^{a_k t}$ (see (2.6)). The missing constants are thus d_{kki}, $i = 0, \ldots, k-1$, $k \ge 1$. All this shows that the problem is the same for each power of $\bar{x}$: for any $i = 0, 1, \ldots$, we have

$$d_{iii} \;=\; 1 \tag{2.8a}$$

$$d_{kji} \;=\; \frac{b_k}{a_j - a_k} d_{k-1,j,i}, \qquad i \le j < k \tag{2.8b}$$

$$\sum_{j=i}^{k} d_{kji} \;=\; 0, \qquad 0 \le i < k. \tag{2.8c}$$

(The last equality comes from the initial condition $m_k(0) = \bar{x}^k$.) Hence, the problem of finding $\{d_{kj0}; 0 \le j \le k\}$ when $\bar{x} \neq 0$ is the same as finding $\{d_{kj}; 0 \le j \le k\}$ when $\bar{x} = 0$. The latter was done in Dufresne (1989), for the special case $b_k = k$. The same arguments work, however, for a general sequence $\{b_k; k \ge 1\}$. The main argument is that

$$\sum_{j=0}^{k} \prod_{\substack{\ell=0 \\ \ell \neq j}}^{k} \frac{1}{a_j - a_\ell} \;=\; 0 \tag{2.9}$$

for any distinct numbers $\alpha_0, \ldots, \alpha_k$. We recall the proof given in Dufresne (1989). Lagrange's formula for partial fractions decomposition of a rational function is

$$\frac{P(x)}{Q(x)} \;=\; \sum_{j=0}^{k} \frac{1}{x - a_j} \frac{P(a_j)}{Q'(a_j)},$$

for polynomials P and Q, the latter with $k+1$ distinct zeros $\{a_0,\ldots,a_k\}$, the former of degree less than or equal to k. Let

$$Q(x) \;=\; \prod_{j=0}^{k}(x-a_j), \qquad P(x) \;=\; \frac{y-x}{Q(y)}.$$

Then

$$P(x) \;=\; \sum_{j=0}^{k}\frac{y-a_j}{Q(y)}\,\frac{Q(x)}{(x-a_j)Q'(a_j)}.$$

Letting $y\to x$ yields (2.9).

For $i=j=0$, we find (from (2.8a,b))

$$d_{k00} \;=\; \left(\prod_{m=1}^{k} b_m\right)\prod_{\substack{\ell=0\\ \ell\neq 0}}^{k}\frac{1}{a_0-a_\ell}, \qquad k\geq 0.$$

Next, $d_{110}=-d_{100}=b_1/(a_1-a_0)$ (from (2.8c)), which implies

$$d_{k10} \;=\; \frac{b_k\cdots b_2}{(a_1-a_k)\cdots(a_1-a_2)}d_{110} \;=\; \left(\prod_{m=1}^{k} b_m\right)\prod_{\substack{\ell=0\\ \ell\neq 1}}^{k}\frac{1}{a_1-a_\ell}, \qquad k\geq 1.$$

To prove the result in general, represent the $\{d_{kj0}\}$ on a grid, with k horizontal and j vertical. For some integer $J>0$, suppose that

$$d_{kj0} \;=\; \left(\prod_{m=1}^{k} b_m\right)\prod_{\substack{\ell=0\\ \ell\neq j}}^{k}\frac{1}{a_j-a_\ell}, \qquad k\geq j, \quad j=0,\ldots,J-1, \tag{2.10}$$

that is, assume the result holds for the lines $j=0,\ldots,J-1$. Then, by (2.8c) and (2.9), (2.10) also holds for $k=j=J$. From this and (2.8b), we find, for $k>J$,

$$d_{kJ0} \;=\; \frac{b_k\cdots b_{J+1}}{(a_J-a_k)\cdots(a_J-a_{J+1})}d_{JJ0} \;=\; \left(\prod_{m=1}^{k} b_m\right)\prod_{\substack{\ell=0\\ \ell\neq J}}^{k}\frac{1}{a_J-a_\ell}.$$

By induction, (2.10) applies for all $0\leq j\leq k$. For the coefficients of $\bar{x}^i, i\geq 1$, the problem is the same, except that it "starts" on line i, so that, in the above arguments, (j,k) has to be replaced with $(j+i,k+i)$. □

Remark. Similar formulas hold when some of the $\{a_0,\ldots,a_K\}$ are identical. The only difference is that terms of the form $t^\ell e^{a_j t}$ make their appearance, where ℓ is an integer which depends on the number of times the same a_k appears in $\{a_0, \ldots, a_K\}$. For instance, suppose $a_0 = 0, a_1 = a_2 \neq 0$. The solution of (2.2) is then

$$m_2(t) \;=\; \bar{x}^2 e^{a_1 t} + b_2\left(\bar{x} + \frac{b_1}{a_1}\right) t e^{a_1 t} - \frac{b_1 b_2}{a_1^2}(e^{a_1 t} - 1).$$

The $te^{a_1 t}$ reappears in $m_j(t)$, $j \geq 3$. The solutions of the differential equations nevertheless depend continuously on the parameters $a_0, a_1, \ldots$, so that the solutions when some of them are identical can be found by taking limits in the solutions given in the theorem. □

As a first application of Theorem 2.1, we generalize Proposition 2 of Dufresne (1989). Suppose $\{S_t\}$ satisfies

$$dS_t \;=\; (\eta S_t + \zeta)\,dt + \sigma S_t\,dW_t, \qquad S_0 \;=\; \bar{x}. \tag{2.11}$$

Then the moments $m_k(t) = \mathsf{E}S_t^k$, $k = 0, 1, \ldots$, satisfy Eqs.(2.3-2.4), with

$$a_k \;=\; k\eta + k(k-1)\sigma^2/2, \qquad b_k \;=\; k\zeta. \tag{2.12}$$

Corollary 2.2. *If the $\{a_k\}$ are all distinct, the moments of the process in (2.11) are given by (2.4), with $\{a_k\}$ as in (2.12) and*

$$d_{kj} \;=\; k!\sum_{i=0}^{j} \frac{\bar{x}^i \zeta^{k-i}}{i!} \prod_{\substack{\ell=i \\ \ell\neq j}}^{k} \frac{1}{a_j - a_\ell}, \qquad j = 0, \ldots, k.$$

Remark. The process S in (2.11) has been called the Courtadon interest rate model (Courtadon, 1982). The moments of the spot rate are thus given by Corollary 2.2. The limit distribution of S_t as t tends to infinity (inverse Gamma if $\eta < \sigma^2/2$, ∞ otherwise) is derived in Dufresne (1990). In that paper, S_t represents the market value of a an initial amount S_0 and of a continuous payment stream of ζ units, all invested in a security represented by a geometric Brownian motion. □

Theorem 2.3. *Suppose $\alpha \neq 0$. If X_t satisfies (1.1), then its moments are*

$$\mathsf{E}X_t^k \;=\; \sum_{j=0}^{k} \theta_{kj} e^{\alpha j t}, \qquad k \;=\; 0, 1, \ldots,$$

where

$$\theta_{kj} \;=\; \sum_{i=0}^{j} \bar{x}^i \frac{k!(-1)^{k-j}\bar{u}^{k-i}}{i!(j-i)!(k-j)!}\frac{(\bar{v})_k}{(\bar{v})_i}, \qquad 0 \leq j \leq k$$

$$\bar{u} \;=\; \frac{\gamma^2}{2\alpha}, \qquad \bar{v} \;=\; \frac{2\beta}{\gamma^2}$$

$$(y)_0 \;=\; 1, \quad (y)_k \;=\; y(y+1)\cdots(y+k-1), \quad k \geq 1.$$

Proof. We have

$$\begin{aligned}\prod_{m=1}^{k} b_m &= k!\left[\frac{\gamma^2}{2}k+\left(\beta-\frac{\gamma^2}{2}\right)\right]\cdots\left[\frac{\gamma^2}{2}1+\left(\beta-\frac{\gamma^2}{2}\right)\right]\\ &= k!\left(\frac{\gamma^2}{2}\right)^k\left[\frac{2\beta}{\gamma^2}+k-1\right]\cdots\frac{2\beta}{\gamma^2}\\ &= k!\left(\frac{\gamma^2}{2}\right)^k(\bar{v})_k.\end{aligned}$$

For $i \le j$,

$$\begin{aligned}\prod_{\substack{\ell=i\\ \ell\neq j}}^{k}(a_j-a_\ell) &= \alpha^{k-i}\prod_{\ell=i}^{j-1}(j-\ell)\prod_{\ell=j+1}^{k}(j-\ell)\\ &= (-1)^{k-j}\alpha^{k-i}(j-i)!(k-j)!.\end{aligned}$$

Hence

$$\left(\prod_{m=i+1}^{k} b_m\right)\prod_{\substack{\ell=i\\ \ell\neq j}}^{k}\frac{1}{a_j-a_\ell} = \frac{k!}{i!}\left(\frac{\gamma^2}{2\alpha}\right)^{k-i}\frac{(\bar{v})_k}{(\bar{v})_i}\frac{(-1)^{k-j}}{(j-i)!(k-j)!}. \qquad \square$$

The MGF of X_t will now be obtained by summing the moments of X_t. We begin by assuming that the following sum converges:

$$\sum_{k=0}^{\infty}\frac{s^k}{k!}\mathsf{E}X_t^k = \sum_{j=0}^{\infty}e^{a_j t}\sum_{k=j}^{\infty}\frac{s^k}{k!}\theta_{kj} = \sum_{k=0}^{\infty}\frac{s^k}{k!}\sum_{j=0}^{k}e^{a_j t}\sum_{i=0}^{j}\bar{x}^i\frac{k!(-1)^{k-j}\bar{u}^{k-i}}{i!(j-i)!(k-j)!}\frac{(\bar{v})_k}{(\bar{v})_i}. \qquad (2.13)$$

This sum will be evaluated by summing first over k, then over j, and finally over i. This procedure needs to be justified, as the summands are not all of the same sign. The justification is given below. We will use the formula

$$(1-y)^{-c} = \sum_{n=0}^{\infty}(c)_n\frac{y^n}{n!}, \qquad (2.14)$$

which is valid for $c \in \mathbb{R}$, $|y| < 1$.

For fixed $i \le j$ and small enough $|s|$, we have

$$\sum_{k=j}^{\infty}\bar{x}^i\frac{e^{a_j t}s^k}{k!}\frac{k!(-1)^{k-j}\bar{u}^{k-i}}{i!(j-i)!(k-j)!}\frac{(\bar{v})_k}{(\bar{v})_i} = \bar{x}^i\frac{(\bar{v})_j}{(\bar{v})_i}\frac{s^j\bar{u}^{j-i}e^{a_j t}}{i!(j-i)!}(1+s\bar{u})^{-\bar{v}-j}.$$

Next,

$$
\begin{aligned}
\bar{x}^i \sum_{j=i}^{\infty} \frac{(\bar{v})_j}{(\bar{v})_i} \frac{s^j \bar{u}^{j-i} e^{a_j t}}{i!(j-i)!} (1+s\bar{u})^{-\bar{v}-j} \\
&= \frac{s^i \bar{x}^i e^{\alpha i t}}{i!} (1+s\bar{u})^{-\bar{v}-i} \sum_{j=i}^{\infty} \frac{(\bar{v})_j}{(\bar{v})_i} \frac{s^{j-i} \bar{u}^{j-i} e^{\alpha t (j-i)}}{(j-i)!} (1+s\bar{u})^{i-j} \\
&= \frac{s^i \bar{x}^i e^{\alpha i t}}{i!} (1+s\bar{u})^{-\bar{v}-i} \left(1 - \frac{s\bar{u}e^{\alpha t}}{1+s\bar{u}}\right)^{-\bar{v}-i} \\
&= \frac{s^i \bar{x}^i e^{\alpha i t}}{i!} \left[1 - s\bar{u}(e^{\alpha t}-1)\right]^{-\bar{v}-i}.
\end{aligned}
$$

Finally, summing over i,

$$
\mathsf{E} e^{sX_t} = \left[1 - s\bar{u}\left(e^{\alpha t}-1\right)\right]^{-\bar{v}} \exp\left[\frac{s\bar{x}e^{\alpha t}}{1 - s\bar{u}\left(e^{\alpha t}-1\right)}\right] = \phi(s)^{\bar{v}} \exp\left[\lambda_t\left(\phi(s)-1\right)\right]. \quad (2.15)
$$

Here

$$
\phi(s) = (1 - s\mu_t)^{-1}, \qquad \mu_t = \frac{\gamma^2}{2}\left(\frac{e^{\alpha t}-1}{\alpha}\right), \qquad \lambda_t = \frac{2\alpha\bar{x}}{\gamma^2(1-e^{-\alpha t})}.
$$

The function $\phi(s)$ is the MGF of an exponential distribution, with mean equal to μ_t. The distribution of X_t is thus the distribution of $X_t' + X_t''$, where the two variables are independent, $X_t' \sim \mathbf{Gamma}(\bar{v}, \mu_t)$, and X_t'' is compound Poisson

$$
X_t'' = \sum_{i=1}^{N} U_i,
$$

with $N \sim \mathbf{Poisson}(\lambda_t)$, $U_i \sim \mathbf{Exp}(\mu_t)$.

We now show that the sum in (2.13) converges. What we have just done is to sum the quantities

$$
c_{ijk} = \frac{s^k \bar{x}^i (-1)^{k-j} \bar{u}^{k-i} e^{a_j t} (\bar{v})_k}{i!(j-i)!(k-j)!(\bar{v})_i}, \qquad 0 \le i \le j \le k < \infty.
$$

The same steps show that

$$
\sum_{0 \le i \le j \le k < \infty} |c_{ijk}| = \left[1 - s\bar{u}\left(e^{\alpha t}+1\right)\right]^{-\bar{v}} \exp\left[\frac{s\bar{x}e^{\alpha t}}{1 - s\bar{u}\left(e^{\alpha t}+1\right)}\right],
$$

if $0 \le |s| < 1/[\bar{u}\,(e^{\alpha t}+1)]$. Therefore: (1) the sum in (2.13) is convergent, (2) the order of summation of the c_{ijk} is irrelevant, and (3) the result is an analytic function of s, at least for $0 \le |s| < 1/[\bar{u}\,(e^{\alpha t}+1)]$. Expression (2.15) is therefore correct for $s < 1/[\bar{u}\,(e^{\alpha t}-1)]$, by analytic continuation.

Obviously, the limit distribution of X_t as $t \to \infty$ exists if, and only if, $\alpha < 0$, in which case it is $\mathbf{Gamma}(2\beta/\gamma^2, -\gamma^2/(2\alpha))$ (the compound Poisson part, which depends on the initial condition $\bar{x}$, disappears in the limit). Finally, the density of X_t may be obtained by conditioning on N. To simplify the algebra, consider $U = X_t/\mu_t$: with probability $e^{-\lambda_t}\lambda_t^n/n!$, the distribution of U is $\mathbf{Gamma}(\bar{v}+n, 1)$, for $n = 0, 1 \ldots$. Hence, the density of U is

$$\begin{aligned} f_U(y) &= \sum_{n=0}^{\infty} e^{-\lambda_t} \frac{\lambda_t^n}{n!} \frac{y^{\bar{v}+n-1}}{\Gamma(\bar{v}+n)} e^{-y} \mathbf{1}_{\{y>0\}} = \left(\frac{y}{\lambda_t}\right)^{\frac{\bar{v}-1}{2}} e^{-\lambda_t - y} \sum_{n=0}^{\infty} \frac{(\sqrt{\lambda_t y})^{\bar{v}-1+2n}}{n!\Gamma(\bar{v}+n)} \mathbf{1}_{\{y>0\}} \\ &= \left(\frac{y}{\lambda_t}\right)^{\frac{\bar{v}-1}{2}} e^{-\lambda_t - y} I_{\bar{v}-1}(2\sqrt{\lambda_t y}) \mathbf{1}_{\{y>0\}}, \end{aligned}$$

where I_ν is the modified Bessel function of the first kind of argument ν (Lebedev, 1972, p.108):

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \qquad |\arg z| < \pi, \qquad \nu \in \mathbb{C}. \tag{2.16}$$

The density of X_t is thus

$$f_{X_t}(x) = \frac{1}{\mu_t} \left(\frac{x e^{-\alpha t}}{\bar{x}}\right)^{\frac{\bar{v}-1}{2}} e^{-\lambda_t - x/\mu_t} I_{\bar{v}-1}\left(\frac{4\alpha}{\gamma^2(e^{\alpha t}-1)} \sqrt{x \bar{x} e^{\alpha t}}\right) \mathbf{1}_{\{x>0\}}. \tag{2.17}$$

(When $\bar{x} = 0$ this reduces to the $\mathbf{Gamma}(\bar{v}, \mu_t)$ density.)

The MGF and the density above may also be obtained from the relationship between the solution of the SDE (1.1) and Bessel processes (see Revuz & Yor, 1999, Chapter 11). For a constant $p > 0$, the process $\tilde{X}_t = pX_t$ satisfies

$$\tilde{X}_t = p\bar{x} + \int_0^t (\alpha \tilde{X}_s + \beta p) ds + \gamma\sqrt{p} \int_0^t \sqrt{\tilde{X}_s} dW_s$$

Letting $p = 4/\gamma^2$, we find that

$$\tilde{X}_t = \tilde{x} + \int_0^t (\alpha \tilde{X}_s + \tilde{\beta}) ds + 2 \int_0^t \sqrt{\tilde{X}_s} dW_s,$$

where $\tilde{x} = 4\bar{x}/\gamma^2$, $\tilde{\beta} = 4\beta/\gamma^2$. Now, consider the square of a δ-dimensional Bessel process, that is, the unique strong solution of

$$Z_t = Z_0 + \delta t + 2 \int_0^t \sqrt{Z_s} dB_s$$

(B standard Brownian motion), and let $g(t) = (1 - e^{-\alpha t})/\alpha$ (see Revuz & Yor, 1999, Exercise (1.13), p.448). Integrating by parts, and using the time change formula for stochastic integrals (Revuz & Yor, 1999, p.180), we find

$$\begin{aligned} e^{\alpha t} Z_{g(t)} &= e^{\alpha t} \left[Z_0 + \delta g(t) + 2 \int_0^t \sqrt{Z_{g(s)}} dB_{g(s)} \right] \\ &= Z_0 + \alpha \int_0^t e^{\alpha s} Z_{g(s)} ds + \delta \int_0^t e^{\alpha s} dg(s) + 2 \int_0^t \sqrt{e^{\alpha s} Z_{g(s)}} e^{\alpha s/2} dB_{g(s)}. \end{aligned}$$

The process

$$\tilde{B}_t \;=\; \int_0^t e^{\alpha s/2}\, dB_{g(s)}$$

is a continuous martingale with quadratic variation equal to t, and is thus standard Brownian motion (with respect to its own filtration). Hence, the process $\tilde{Z}_t = e^{\alpha t} Z_{g(t)}$ satisfies

$$\tilde{Z}_t \;=\; Z_0 + \int_0^t (\alpha \tilde{Z}_s + \delta)\, dt + 2\int_0^t \sqrt{\tilde{Z}_s}\, d\tilde{B}_s.$$

Consequently, the processes $\{\tilde{Z}_t; t \geq 0\}$ and $\{\tilde{X}_t; t \geq 0\}$ have the same distribution, if

$$Z_0 \;=\; \tilde{x} \;=\; \frac{4\bar{x}}{\gamma^2}, \qquad \delta \;=\; \tilde{\beta} \;=\; \frac{4\beta}{\gamma^2}.$$

Finally, the process $\{X_t; t \geq 0\}$ has the same distribution as $\{\frac{\gamma^2}{4} e^{\alpha t} Z_{g(t)}; t \geq 0\}$. The MGF of Z_τ is (Revuz & Yor, 1999, p.441)

$$\mathsf{E}\, e^{\rho Z_\tau} \;=\; (1 - 2\rho\tau)^{-\delta/2} \exp\left[\rho Z_0/(1 - 2\rho\tau)\right].$$

It can be checked that appropriate substitutions yield (2.15). In the same fashion, the density of Z_τ is

$$f_{Z_\tau}(z) \;=\; \frac{1}{2\tau}\left(\frac{z}{Z_0}\right)^{\frac{1}{2}\left(\frac{\delta}{2}-1\right)} e^{-(Z_0+z)/2\tau} I_{\frac{\delta}{2}-1}\left(\frac{\sqrt{Z_0 z}}{\tau}\right) \mathbf{1}_{\{z>0\}}.$$

This implies that the density of X_t is (for $x > 0$)

$$\frac{2\alpha e^{-\alpha t}}{\gamma^2(1 - e^{-\alpha t})}\left(\frac{x e^{-\alpha t}}{\bar{x}}\right)^{\frac{1}{2}(\bar{v}-1)} \exp\left[-\left(\frac{4\bar{x}}{\gamma^2} + \frac{4x e^{-\alpha t}}{\gamma^2}\right)\alpha/[2(1 - e^{-\alpha t})]\right] I_{\bar{v}-1}\left(\frac{4\alpha\sqrt{x\bar{x}e^{-\alpha t}}}{\gamma^2(1 - e^{-\alpha t})}\right)$$

$$= \frac{1}{\mu_t}\left(\frac{x e^{-\alpha t}}{\bar{x}}\right)^{\frac{1}{2}(\bar{v}-1)} e^{-\lambda_t - x/\mu_t} I_{\bar{v}-1}\left(\frac{4\alpha\sqrt{x\bar{x}e^{-\alpha t}}}{\gamma^2(1 - e^{-\alpha t})}\right),$$

which agrees with (2.17).

It is also possible to find an expression for $\mathsf{E}\, X_t^p$ for non-integral p. One could integrate the density times x^p term by term; an alternative is the following: since

$$\frac{1}{y^q} \;=\; \frac{1}{\Gamma(q)}\int_0^\infty s^{q-1} e^{-sy}\, ds, \qquad q, y > 0, \tag{2.18}$$

we have

$$\begin{aligned}
\mathsf{E}\, X_t^{-q} &= \frac{1}{\Gamma(q)}\int_0^\infty s^{q-1}\, \mathsf{E}\, e^{-sX_t} ds \\
&= \frac{1}{\Gamma(q)}\int_0^\infty s^{q-1} \phi(-s)^{\bar{v}} e^{\lambda_t(\phi(-s)-1)} ds \\
&= \frac{\mu_t^{-q} e^{-\lambda_t}}{\Gamma(q)}\int_0^1 r^{\bar{v}-q-1}(1-r)^{q-1} e^{\lambda_t r}\, dr \qquad (2.19) \\
&= \mu_t^{-q} e^{-\lambda_t} \frac{\Gamma(\bar{v}-q)}{\Gamma(\bar{v})}\, {}_1F_1(\bar{v}-q, \bar{v}; \lambda_t), \qquad (2.20)
\end{aligned}$$

where ${}_1F_1$ is the confluent hypergeometric function

$$ {}_1F_1(r,s;z) \;=\; \sum_{n=0}^{\infty} \frac{(r)_n}{(s)_n}\frac{z^n}{n!}, \qquad r,s,z \in \mathbb{C}, \quad s \neq 0,-1,\ldots $$

Here we have used the integral representation (9.11.1), p.266 of Lebedev (1972) for the confluent hypergeometric function. The above calculation is seen to be correct for $0 \le q < \bar{v}$, while $\mathsf{E}X_t^{-q}$ is infinite for $q \le \bar{v}$ (this is because the integral in (2.19) tends to infinity as $q \uparrow \bar{v}$). Since $\mathsf{E}X_t^{-q}$ is an analytic function of q, and (2.20) is analytic for $\mathrm{Re}(-q) > -\bar{v}$, we get the following result.

Theorem 2.4. $\mathsf{E}X_t^p \;=\; \mu_t^p e^{-\lambda_t}\dfrac{\Gamma(\bar{v}+p)}{\Gamma(\bar{v})}\,{}_1F_1(\bar{v}+p,\bar{v};\lambda_t), \qquad p > -2\beta/\gamma^2.$

This formula may be reconciled with the one in Theorem 2.3 by appealing to the identity (Lebedev, 1972, p.267)

$$ {}_1F_1(r,s;z) \;=\; e^z\,{}_1F_1(s-r,s;-z), \qquad s \neq 0,-1,\ldots $$

If $p = k$, a non-negative integer, then $\mathsf{E}X_t^k$ may be expressed as a series which terminates:

$$ \begin{aligned} \mathsf{E}X_t^k \;&=\; \mu_t^k \frac{\Gamma(\bar{v}+k)}{\Gamma(\bar{v})}\,{}_1F_1(-k,\bar{v};-\lambda_t) \\ &=\; \sum_{i=0}^{k} \frac{(-1)^i(-k)_i\,(\bar{v})_k}{(\bar{v})_i}\frac{\mu_t^i\lambda_t^i}{i!}\mu_t^{k-i} \\ &=\; \sum_{i=0}^{k} \frac{k!(\bar{v})_k}{(k-i)!(\bar{v})_i}\frac{\bar{x}^i e^{\alpha i t}}{i!}\bar{u}^{k-i}\sum_{\ell=0}^{k-i}\binom{k-i}{\ell}e^{\alpha\ell t}(-1)^{k-i-\ell} \\ &=\; \sum_{i=0}^{k} \frac{k!\,(\bar{v})_k}{(k-i)!(\bar{v})_i}\frac{\bar{x}^i e^{\alpha i t}}{i!}\bar{u}^{k-i}\sum_{j=i}^{k}\frac{(k-i)!}{(j-i)!(k-j)!}e^{\alpha(j-i)t}(-1)^{k-j} \\ &=\; \sum_{j=0}^{k} e^{\alpha j t}\sum_{i=0}^{j}\bar{x}^i\frac{k!(-1)^{k-j}\bar{u}^{k-i}(\bar{v})_k}{i!(j-i)!(k-j)!(\bar{v})_i}. \end{aligned} $$

3. Moments of the integrated square-root process

In this section, two techniques are given for the calculation of the moments of the integral of the square-root process $\{Y_t\}$.

Suppose X is the solution of SDE (1.1), and define

$$ Y_t \;=\; \int_0^t X_s\,ds, \qquad M_{jk}(t) \;=\; \mathsf{E}Y_t^j X_t^k, $$

for non-negative integers j,k; M_{jk} is finite for all $t \geq 0$, since Y_t has finite moments of all order (since the Laplace transform of Y_t exists in a neighbourhood of the origin, see Section 4). Applying Ito's formula, we get

$$\begin{aligned} d(Y_t^j X_t^k) &= X_t^k\, d(Y_t^j) + Y_t^j\, d(X_t^k) \\ &= jY_t^{j-1}X_t^{k+1}\, dt + Y_t^j(a_k X_t^k + b_k X_t^{k-1})\, dt + \gamma k Y_t^j X_t^{k-\frac{1}{2}}\, dW_t. \end{aligned}$$

Since $\mathsf{E} X_t^p < \infty$ for all positive p, it follows that

$$\mathsf{E}\int_0^t \left(\gamma k Y_s^j X_s^{k-\frac{1}{2}}\right)^2 ds \;<\; \infty$$

for all $j,k \in \mathbb{N}$, and so

$$\mathsf{E}\int_0^t \gamma k Y_s^j X_s^{k-\frac{1}{2}}\, dW_s \;=\; 0.$$

This implies

$$\frac{d}{dt}M_{jk}(t) \;=\; a_k M_{jk}(t) + b_k M_{j,k-1}(t) + jM_{j-1,k+1}(t). \tag{3.1}$$

These are linear ordinary differential equations with constant coefficients, and their solutions are straightforward (though tedious to obtain by hand). In order to find $M_{j0}(t)$, it is necessary to calculate $M_{j-1,1}(t)$, $M_{j-2,2}(t)$, and so on. If we represent the $\{M_{jk}\}$ on a grid, with j horizontal and k vertical, M_{j0} is obtained after solving the differential equations on the diagonals $M_{0,i},\ldots,M_{i,0}$ for $i \leq j$. The following result is proved by suitably adapting the arguments used in Section 2.

Theorem 3.1. *Suppose $\alpha \neq 0$, and that X satisfies (1.1). Then, for $j,k \in \mathbb{N}$,*

$$M_{jk}(t) \;=\; \sum_{m=0}^{j+k} M_{jkm}(t)e^{\alpha m t}, \tag{3.2}$$

where $M_{jkm}(t)$ is a polynomial in t with degree

$$\deg M_{jkm} \;\leq\; \begin{cases} j & \text{if}\quad 0 \leq m \leq k \\ j+k-m & \text{if}\quad k+1 \leq m \leq j+k. \end{cases} \tag{3.3}$$

Proof. We proceed by induction on j and then on k. The theorem is correct for $j=0$ and all $k \geq 0$ by Theorem 2.3. Suppose (3.2)-(3.3) are correct for $j = 0,\ldots,J-1$ and all $k \geq 0$, for some $J \geq 1$. Based on this assumption, the solution of (3.1) is

$$M_{Jk} \;=\; e^{a_k t}M_{Jk}(0) + e^{a_k t}\int_0^t e^{-a_k s}R_{Jk}(s)\, ds, \tag{3.4}$$

where

$$\begin{aligned} R_{Jk}(t) &= b_k M_{J,k-1}(t) + JM_{J-1,k+1}(t) \\ &= \sum_{m=0}^{J+k-1} [b_k M_{J,k-1,m}(t) + JM_{J-1,k+1,m}(t)]e^{a_m t} + JM_{J-1,k+1,J+k}(t)e^{a_{J+k}t}. \end{aligned}$$

We show that this implies that (3.2)-(3.3) hold for $j = J, k = 0$. Since $a_0 = b_0 = M_{J0}(0) = 0$, (3.4) reduces to

$$M_{J0}(t) \;=\; J \sum_{m=0}^{J} \int_0^t M_{J-1,1,m}(s) e^{a_m s}\, ds. \tag{3.5}$$

Consider the terms corresponding to $m = 1, \ldots, J$ in the sum above. Recall that $a_m = \alpha m \neq 0$ and that, for any constant $A \neq 0$,

$$\int_0^t e^{As} s^n \, ds$$

is the sum of a constant and of a polynomial of degree n times e^{At}. Hence

$$\int_0^t M_{J-1,1,m}(s)\, e^{a_m s}\, ds$$

is the sum of a constant and of a polynomial of the same degree as $M_{J-1,1,m}$ times $e^{a_m t}$.

Since $a_0 = 0$, he first term on the right hand side of (3.5) is seen to be a polynomial of degree equal to one plus the degree of $M_{J-1,1,0}$.

The above considerations imply that

$$M_{J0}(t) \;=\; \sum_{m=0}^{J} M_{J0m}(t) e^{\alpha m t},$$

where the functions $M_{J0m}(t)$ are polynomials in t. Moreover, the induction assumption says in particular that

$$\deg M_{J-1,1,m}(t) \;\leq\; \begin{cases} J-1 & \text{if} \;\; 0 \leq m \leq 1 \\ J-m & \text{if} \;\; 2 \leq m \leq J \end{cases}$$

and, consequently, what we have just seen also means that

$$\deg M_{J0m} \;\leq\; \begin{cases} J & \text{if} \;\; m = 0 \\ J-m & \text{if} \;\; 1 \leq m \leq J. \end{cases}$$

Keeping J fixed, make another induction assumption: (3.2)-(3.3) hold for $j = J$ and $k = 0, \ldots, K-1$, for some $K \geq 1$. The proof will be finished once we show that (3.2)-(3.3) hold for $j = J$ and $k = K$ as well. By the induction assumptions and (3.4),

$$\begin{aligned} M_{JK}(t) \;=\; e^{a_K t} \Bigg\{ & \sum_{m=0}^{J+K-1} \int_0^t [b_k M_{J,K-1,m}(s) + J M_{J-1,K+1,m}(s)] e^{(am - a_K)s}\, ds \\ & + \int_0^t J M_{J-1,K+1,J+K}(s) e^{(a_{J+K} - a_K)s}\, ds \Bigg\}. \end{aligned} \tag{3.6}$$

By the same reasoning as before, we see that M_{JK} is of the form

$$M_{JK}(t) \;=\; \sum_{m=0}^{J+K} M_{JKm}(t) e^{a_m t},$$

where $M_{JKm}(t)$ is a polynomial. In (3.6), the terms of the sum inside the curly brackets with $m \neq K$ reduce to

$$\text{constant} + \text{polynomial} \times e^{(a_m - a_K)t}$$

The degree of the polynomial is the same as that of

$$b_k M_{J,K-1,m}(s) + J M_{J-1,K+1,m}(s),$$

that is, no larger than

$$\max[\min(J, J+K-1-m), \min(J-1, J+K-m)] \;=\; \min(J, J+K-m). \tag{3.7}$$

The same applies to the term following the sum, which corresponds to $m = J+K$. For the term $m = K$ inside the curly brackets, the integral reduces to a polynomial of degree no larger than

$$1 + \max[\deg(M_{J,K-1,K}), \deg(M_{J-1,K+1,K}) \;=\; 1 + \max(J-1, J-1) \;=\; J. \tag{3.8}$$

All these integrals are then multiplied by $e^{a_K t}$, so that (3.7) becomes the degree of M_{JKm} for $m \neq K$, and (3.8) becomes the degree of M_{JKK}. This ends the proof. $\square$

A general simplified formula for the polynomials M_{jkm} has not been found, but they can be calculated recursively, as the next theorem shows.

Theorem 3.2. *If we let*

$$\begin{aligned} M_{jkm}(t) &= \sum_{n=0}^{j \wedge (j+k-m)} M_{jkmn} t^n, \\ R_{jkmn} &= b_k M_{j,k-1,m,n} + j M_{j-1,k+1,m,n}, \end{aligned}$$

then

$$M_{jkmn} \;=\; -\sum_{i=n}^{j\wedge(j+k-m)} \frac{(n+1)_{i-n}}{[\alpha(k-m)]^{i-n+1}} R_{jkmi}, \qquad k \neq m \tag{3.9}$$

$$M_{jkkn} \;=\; -\frac{1}{n} R_{j,k,k,n-1}, \qquad n = 1, \ldots, j \tag{3.10}$$

$$M_{jkk0} \;=\; -\sum_{\substack{m=0 \\ m\neq k}}^{j+k} M_{jkm0}, \qquad j \geq 1 \tag{3.11}$$

$$M_{0km0} \;=\; \theta_{km} \qquad \forall\ (k,m) \quad (\textit{see Theorem 2.3}) \tag{3.12}$$

$$M_{0kmn} \;=\; 0, \qquad n \geq 1. \tag{3.13}$$

Proof. By Theorem 3.1, the derivative on the left hand side of Eq.(3.1) equals

$$\frac{d}{dt}\sum_{m=0}^{j+k} e^{a_m t}\sum_{n=0}^{j} M_{jkmn}t^n \;=\; \sum_{m=0}^{j+k} e^{a_m t}\sum_{n=0}^{j}[a_m M_{jkmn}+(n+1)M_{j,k,m,n+1}]t^n.$$

Here, in order to simplify the notation, we have added some M_{jkmn} which are necessarily 0, since $n > \min(j, j+k-m)$. The right hand side of (3.1) may be written as

$$\sum_{m=0}^{j+k} e^{a_m t}\sum_{n=0}^{j}[a_k M_{jkmn}+b_k M_{j,k-1,m,n}+jM_{j-1,k+1,m,n}]t^n.$$

Hence,

$$M_{j,k,m,n+1} \;=\; \frac{1}{n+1}\left[(a_k-a_m)M_{jkmn}+R_{jkmn}\right]. \tag{3.14}$$

If $k=m$, then this is (3.10). If $k\neq m$, it can be verified that (3.9) is the solution of (3.14) that satisfies the condition

$$M_{j,k,m,1+j\wedge(j+k-m)} \;=\; 0$$

(which must hold by Theorem 3.1). Eq.(3.11) results from $M_{jk}(0)=0$ for any $j\geq 1$, while (3.12)-(3.13) follow from Theorem 2.3. $\square$

Eqs.(3.9)-(3.12) are easy to program, and give the explicit formulas for any moment of Y_t much faster than solving the differential equations (3.1). The recursion must proceed through the values $(j,k)=(0,0),(0,1),(1,0),(0,2),(1,1),(2,0)$, and so on, up to the highest required moment $\mathsf{E}Y_t^J = M_{J0}(t)$. Observe that those formulas only need to be generated symbolically once. The first three moments of Y_t are:

$$\begin{aligned}
\mathsf{E}Y_t \;&=\; -\frac{\bar{x}}{\alpha}-\frac{\beta}{\alpha^2}-\frac{t\beta}{\alpha}+e^{t\alpha}\Big(\frac{\bar{x}}{\alpha}+\frac{\beta}{\alpha^2}\Big)\\
\mathsf{E}Y_t^2 \;&=\; \frac{\bar{x}^2}{\alpha^2}+\frac{2\bar{x}\beta}{\alpha^3}+\frac{\beta^2}{\alpha^4}-\frac{\bar{x}\gamma^2}{\alpha^3}-\frac{5\beta\gamma^2}{2\alpha^4}+t\Big(\frac{2\bar{x}\beta}{\alpha^2}+\frac{2\beta^2}{\alpha^3}-\frac{\beta\gamma^2}{\alpha^3}\Big)+\frac{t^2\beta^2}{\alpha^2}\\
&\quad+e^{t\alpha}\Big[-\frac{2\bar{x}^2}{\alpha^2}-\frac{4\bar{x}\beta}{\alpha^3}-\frac{2\beta^2}{\alpha^4}+\frac{2\beta\gamma^2}{\alpha^4}+t\Big(-\frac{2\bar{x}\beta}{\alpha^2}-\frac{2\beta^2}{\alpha^3}-\frac{2\bar{x}\gamma^2}{\alpha^2}-\frac{2\beta\gamma^2}{\alpha^3}\Big)\Big]\\
&\quad+e^{2t\alpha}\Big(\frac{\bar{x}^2}{\alpha^2}+\frac{2\bar{x}\beta}{\alpha^3}+\frac{\beta^2}{\alpha^4}+\frac{\bar{x}\gamma^2}{\alpha^3}+\frac{\beta\gamma^2}{2\alpha^4}\Big)\\
\mathsf{E}Y_t^3 \;&=\; -\frac{\bar{x}^3}{\alpha^3}-\frac{3\bar{x}^2\beta}{\alpha^4}-\frac{3\bar{x}\beta^2}{\alpha^5}-\frac{\beta^3}{\alpha^6}+\frac{3\bar{x}^2\gamma^2}{\alpha^4}+\frac{21\bar{x}\beta\gamma^2}{2\alpha^5}+\frac{15\beta^2\gamma^2}{2\alpha^6}-\frac{3\bar{x}\gamma^4}{\alpha^5}-\frac{11\beta\gamma^4}{\alpha^6}\\
&\quad+t\Big(-\frac{3\bar{x}^2\beta}{\alpha^3}-\frac{6\bar{x}\beta^2}{\alpha^4}-\frac{3\beta^3}{\alpha^5}+\frac{6\bar{x}\beta\gamma^2}{\alpha^4}+\frac{21\beta^2\gamma^2}{2\alpha^5}-\frac{3\beta\gamma^4}{\alpha^5}\Big)\\
&\quad+t^2\Big(-\frac{3\bar{x}\beta^2}{\alpha^3}-\frac{3\beta^3}{\alpha^4}+\frac{3\beta^2\gamma^2}{\alpha^4}\Big)-\frac{t^3\beta^3}{\alpha^3}\\
&\quad+e^{t\alpha}\Big[\frac{3\bar{x}^3}{\alpha^3}+\frac{9\bar{x}^2\beta}{\alpha^4}+\frac{9\bar{x}\beta^2}{\alpha^5}+\frac{3\beta^3}{\alpha^6}-\frac{3\bar{x}^2\gamma^2}{\alpha^4}-\frac{33\bar{x}\beta\gamma^2}{2\alpha^5}-\frac{27\beta^2\gamma^2}{2\alpha^6}-\frac{3\bar{x}\gamma^4}{2\alpha^5}+\frac{15\beta\gamma^4}{2\alpha^6}
\end{aligned}$$

$$
\begin{aligned}
&+t\Big(\frac{6\bar{x}^2\beta}{\alpha^3}+\frac{12\bar{x}\beta^2}{\alpha^4}+\frac{6\beta^3}{\alpha^5}+\frac{6\bar{x}^2\gamma^2}{\alpha^3}+\frac{9\bar{x}\beta\gamma^2}{\alpha^4}-\frac{3\beta^2\gamma^2}{\alpha^5}-\frac{3\bar{x}\gamma^4}{\alpha^4}-\frac{9\beta\gamma^4}{\alpha^5}\Big)\\
&+t^2\Big(\frac{3\bar{x}\beta^2}{\alpha^3}+\frac{3\beta^3}{\alpha^4}+\frac{6\bar{x}\beta\gamma^2}{\alpha^3}+\frac{6\beta^2\gamma^2}{\alpha^4}+\frac{3\bar{x}\gamma^4}{\alpha^3}+\frac{3\beta\gamma^4}{\alpha^4}\Big)\Big]\\
&+e^{2t\alpha}\Big[-\frac{3\bar{x}^3}{\alpha^3}-\frac{9\bar{x}^2\beta}{\alpha^4}-\frac{9\bar{x}\beta^2}{\alpha^5}-\frac{3\beta^3}{\alpha^6}-\frac{3\bar{x}^2\gamma^2}{\alpha^4}+\frac{3\bar{x}\beta\gamma^2}{2\alpha^5}+\frac{9\beta^2\gamma^2}{2\alpha^6}+\frac{3\bar{x}\gamma^4}{\alpha^5}+\frac{3\beta\gamma^4}{\alpha^6}\\
&+t\Big(-\frac{3\bar{x}^2\beta}{\alpha^3}-\frac{6\bar{x}\beta^2}{\alpha^4}-\frac{3\beta^3}{\alpha^5}-\frac{6\bar{x}^2\gamma^2}{\alpha^3}-\frac{15\bar{x}\beta\gamma^2}{\alpha^4}-\frac{15\beta^2\gamma^2}{2\alpha^5}-\frac{6\bar{x}\gamma^4}{\alpha^4}-\frac{3\beta\gamma^4}{\alpha^5}\Big)\Big]\\
&+e^{3t\alpha}\Big(\frac{\bar{x}^3}{\alpha^3}+\frac{3\bar{x}^2\beta}{\alpha^4}+\frac{3\bar{x}\beta^2}{\alpha^5}+\frac{\beta^3}{\alpha^6}+\frac{3\bar{x}^2\gamma^2}{\alpha^4}+\frac{9\bar{x}\beta\gamma^2}{2\alpha^5}+\frac{3\beta^2\gamma^2}{2\alpha^6}+\frac{3\bar{x}\gamma^4}{2\alpha^5}+\frac{\beta\gamma^4}{2\alpha^6}\Big).
\end{aligned}
$$

4. Laplace transform, inverse moments and density of Y_t

In this section and the next one, the symbol $\mathcal{O}$ ("big oh") has the following meaning: we write

$$g(s) \;=\; \mathcal{O}(h(s)) \qquad \text{as} \quad s\to\infty$$

if $|g(s)/h(s)|$ remains bounded as $s\to\infty$. In words, $g(s)$ is big oh of $h(s)$ as s tends to infinity if $|g(s)|$ is no larger than some constant times $|h(s)|$ when s is large. We write

$$g(s) \;=\; o(h(s)) \qquad \text{as} \quad s\to\infty$$

("$g(s)$ is little oh of $h(s)$ as $s\to\infty$") if

$$\lim_{s\to\infty}\frac{g(s)}{h(s)} \;=\; 0.$$

Another symbol used below is "$\sim$", meaning "asymptotically equal to": we write $g(s)\sim h(s)$ if

$$\lim_{s\to\infty}\frac{g(s)}{h(s)} \;=\; 1.$$

Finally, "i" is now the imaginary unit of the complex numbers (that is, $i^2=-1$).

Since the Laplace transform of the integral of the squared Bessel process may be derived explicitly (see Revuz & Yor, 1999, p.445), it is not surprising that the Laplace transform of the square-root process also has an explicit expression. A quick way to find $\mathsf{E}\exp(-sY_t)$ is to suitably modify the CIR formula for the price of a zero-coupon bond (Cox *et al.*, 1985). Suppose the short rate follows a process of type (1.1); the bond price is known to be the expectation of the exponential of minus the integral of the short rate. Multiplying the spot rate by a positive number yields another process which also satisfies (1.1), but with different parameters. Rewriting the bond price formula for this new process immediately yields the Laplace transform of Y_t. The details follow.

Suppose X is the solution of (1.1), let $s>0$, and define

$$\widetilde{X}_t \;=\; sX_t, \qquad \widetilde{X}_0 \;=\; \widetilde{x} \;=\; s\bar{x}.$$

Then

$$\widetilde{X}_t \;=\; \widetilde{x} + \int_0^t (\alpha \widetilde{X}_u + \widetilde{\beta})\,du + \int_0^t \widetilde{\gamma}\sqrt{\widetilde{X}_u}\,dW_u,$$

where $\widetilde{\beta} = \beta s$ amd $\widetilde{\gamma} = \gamma\sqrt{s}$. The formula for the price of a zero-coupon bond maturing in t years is (Cox *et al.*, 1985)

$$\begin{aligned}
\mathsf{E}e^{-sY_t} \;&=\; \mathsf{E}e^{-\int_0^t \widetilde{X}_u\,du} \\
&=\; \left[\frac{2\sqrt{\alpha^2+2\widetilde{\gamma}^2}\,e^{(\sqrt{\alpha^2+2\widetilde{\gamma}^2}-\alpha)t/2}}{(\sqrt{\alpha^2+2\widetilde{\gamma}^2}-\alpha)(e^{\sqrt{\alpha^2+2\widetilde{\gamma}^2}t}-1)+2\sqrt{\alpha^2+2\widetilde{\gamma}^2}}\right]^{\frac{2\tilde{\beta}}{\tilde{\gamma}^2}} \\
&\quad\times \exp\left[-\widetilde{x}\,\frac{2(e^{\sqrt{\alpha^2+2\tilde{\gamma}^2}t}-1)}{(\sqrt{\alpha^2+2\widetilde{\gamma}^2}-\alpha)(e^{\sqrt{\alpha^2+2\tilde{\gamma}^2}}-1)+2\sqrt{\alpha^2+2\widetilde{\gamma}^2}}\right] \\
&=\; \left[\frac{e^{-\alpha t/2}}{\cosh(Pt/2)-\frac{\alpha}{P}\sinh(Pt/2)}\right]^{\frac{2\beta}{\gamma^2}} \exp\left[-\frac{s\bar{x}}{P}\,\frac{2\sinh(Pt/2)}{\cosh(Pt/2)-\frac{\alpha}{P}\sinh(Pt/2)}\right], \quad (4.1)
\end{aligned}$$

where $P = P(s) = \sqrt{\alpha^2+2\gamma^2 s}$.

The Laplace transform of Y_t is finite in a neighborhood of the origin. This is seen by noting that

$$\cosh(Pt/2), \qquad \frac{\alpha}{P}\sinh(Pt/2)$$

are both analytic functions of P, and that their difference does not vanish at $P(0) = |\alpha|$, for any values of the parameters α, β, γ considered, and for any $t > 0$. Consequently, (1) all the moments of Y_t are finite, and (2) they are given by the familiar formula

$$\mathsf{E}Y_t^k \;=\; (-1)^n \frac{d^k}{ds^k}\mathsf{E}e^{-sY_t}\Big|_{s=0}, \qquad k \in \mathbb{N}.$$

The Laplace transform gives a third method of calculating the moments of Y_t. The author has compared the computation times for the moments of Y_t up to order 6 using Mathematica; the recursive method of Section 3 does a lot better than the others, especially for higher moments. As an example, on what is now a rather slow machine (Macintosh 3400 180 Mhz), Mathematica took 377 seconds to find the general symbolic formula for the 6th moment by differentiating the Laplace transform, and a bit more to find the general formulas for moments of order 1 to 6 by solving differential equations (3.1). The recursive method produced all 6 moments in a little more than 8 seconds. A program recursively calculating just the numerical values of the moments (rather than the general symbolic formulas) would of course execute much faster, especially in C.

The next theorem concerns the moments and MGF of $1/Y_t$.

Theorem 4.1. *(a)* $\mathsf{E}Y_t^r$ *is finite for all* $r \in \mathbb{R}$, *and*

$$\mathsf{E}Y_t^{-q} \;=\; \frac{1}{\Gamma(q)}\int_0^\infty s^{q-1}\mathsf{E}\exp(-sY_t)\,ds, \qquad q > 0.$$

(b) $\mathsf{E}\exp\left(\dfrac{p}{Y_t}\right) \;=\; 1+\sqrt{p}\displaystyle\int_0^\infty s^{-\frac{1}{2}} I_1(2\sqrt{ps})\mathsf{E}\,e^{-sY_t}\,ds$ *for all* $p \geq 0.$

This is finite if, and only if, $0 \leq p < \dfrac{1}{2\gamma^2}\,(\beta t+\bar{x})^2.$

(c) $\mathsf{E}\exp\left(p\displaystyle\int_0^t \frac{1}{X_s}\,ds\right) \;=\; \infty$ *for all* $p \geq \dfrac{1}{2\gamma^2}\left(\beta+\dfrac{\bar{x}}{t}\right)^2.$

Proof. By (2.14),

$$P \;=\; \gamma\sqrt{2s}\sqrt{1+\frac{\alpha^2}{2\gamma^2 s}} \;=\; \gamma\sqrt{2s}\left[1+\mathcal{O}\left(s^{-1}\right)\right].$$

From this, we can find the asymptotic behaviour of the first factor in (4.1):

$$\begin{aligned}\left[\frac{e^{-\alpha t/2}}{\cosh(Pt/2)-\frac{\alpha}{P}\sinh(Pt/2)}\right]^{\frac{2\beta}{\gamma^2}} \;&=\; 2^{\frac{2\beta}{\gamma^2}}\,e^{-\alpha\beta t/\gamma^2}\left[\left(1-\frac{\alpha}{P}\right)e^{Pt/2}+\left(1+\frac{\alpha}{P}\right)e^{-Pt/2}\right]^{-\frac{2\beta}{\gamma^2}}\\ &=\; 2^{\frac{2\beta}{\gamma^2}}\,e^{-\alpha\beta t/\gamma^2-\beta Pt/\gamma^2}\left[\left(1-\frac{\alpha}{P}\right)+\left(1+\frac{\alpha}{P}\right)e^{-Pt}\right]^{-\frac{2\beta}{\gamma^2}}\\ &\sim\; 2^{\frac{2\beta}{\gamma^2}}\,e^{-\alpha\beta t/\gamma^2-\beta t\sqrt{2s}/\gamma}\end{aligned}$$

as s tends to infinity. Now turn to the second factor:

$$\begin{aligned}-\frac{s\bar{x}}{P}\frac{2\sinh(Pt/2)}{\cosh(Pt/2)-\frac{\alpha}{P}\sinh(Pt/2)} \;&=\; -\frac{2s\bar{x}}{\sqrt{\alpha^2+2\gamma^2 s}}\,\frac{1}{\coth(Pt/2)-\frac{\alpha}{\sqrt{\alpha^2+2\gamma^2 s}}}\\ &=\; -\frac{\sqrt{2s}\bar{x}}{\gamma}\left(1+\frac{\alpha^2}{2\gamma^2 s}\right)^{-\frac{1}{2}}\frac{1}{\frac{1+e^{-Pt}}{1-e^{-Pt}}-\frac{\alpha}{\gamma\sqrt{2s}}\left(1+\frac{\alpha^2}{2\gamma^2 s}\right)^{-\frac{1}{2}}}\\ &=\; -\frac{\sqrt{2s}\bar{x}}{\gamma}\left[1+\mathcal{O}\left(s^{-1}\right)\right]\frac{1}{1+\mathcal{O}(e^{-Pt})-\frac{\alpha}{\gamma\sqrt{2s}}+\mathcal{O}\left(s^{-\frac{3}{2}}\right)}\\ &=\; -\frac{\sqrt{2s}\bar{x}}{\gamma}\left[1+\mathcal{O}\left(s^{-1}\right)\right]\left[1+\frac{\alpha}{\gamma\sqrt{2s}}+\mathcal{O}\left(s^{-1}\right)\right]\\ &=\; -\frac{\sqrt{2s}\bar{x}}{\gamma}-\frac{\alpha\bar{x}}{\gamma^2}+\mathcal{O}\left(s^{-\frac{1}{2}}\right).\end{aligned}$$

From these expressions we get

$$\mathsf{E}\,e^{-sY_t} \;\sim\; 2^{\frac{2\beta}{\gamma^2}}\exp\left\{-(\beta t+\bar{x})\left(\frac{\alpha}{\gamma^2}+\frac{\sqrt{2s}}{\gamma}\right)\right\} \qquad \text{as} \quad s\to\infty. \tag{4.2}$$

By this asymptotic identity and (2.18), $\mathsf{E}Y_t^{-q}$ is finite if, and only if, the integral

$$2^{\frac{2\beta}{\gamma^2}} e^{-\frac{\alpha}{\gamma^2}(\beta t+\bar{x})} \frac{1}{\Gamma(q)} \int_0^\infty s^{q-1} \exp\left\{-\frac{\sqrt{2s}}{\gamma}(\beta t+\bar{x})\right\} ds,$$

is finite, which is true for all $q>0$. This proves part (a) of the theorem.

Now turn to part (b). For $p \geq 0$,

$$\begin{aligned}
\mathsf{E}\exp\left(\frac{p}{Y_t}\right) &= \sum_{n=0}^{\infty} \frac{p^n}{n!} \mathsf{E}Y_t^{-n} \\
&= 1 + \int_0^\infty \sum_{n=1}^{\infty} \frac{p^n s^{n-1}}{\Gamma(n)\Gamma(n+1)} \mathsf{E}\exp(-sY_t)\, ds \\
&= 1 + \int_0^\infty \sqrt{\frac{p}{s}} \sum_{k=0}^{\infty} \frac{(2\sqrt{ps}/2)^{1+2k}}{\Gamma(k+1)\Gamma(k+2)} \mathsf{E}\exp(-sY_t)\, ds \\
&= 1 + \int_0^\infty \sqrt{\frac{p}{s}}\, I_1(2\sqrt{ps})\, \mathsf{E}\exp(-sY_t)\, ds
\end{aligned}$$

(see the definition of the modified Bessel function I_ν in (2.16)).

It remains to be seen for what $p>0$ the above integral is finite. The integrand is continuous in s, and bounded near the origin. It is known that

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}, \qquad x \to \infty \tag{4.3}$$

(Lebedev, 1972, p.123). Combining this with (4.2), we find that

$$\sqrt{\frac{p}{s}}\, I_1(2\sqrt{ps})\, \mathsf{E}\exp(-sY_t) \sim Cs^{-3/4} \exp\left\{\sqrt{2s}\left[\sqrt{2p} - \frac{1}{\gamma}(\beta t+\bar{x})\right]\right\}$$

as $s \to \infty$ (here $C>0$ depends on p but not on s). The function on the right is integrable over $[1,\infty)$ if, and only if,

$$p < \frac{1}{2\gamma^2}(\beta t+\bar{x})^2.$$

To prove (c), note that by the Cauchy-Schwarz inequality

$$\left[\int_0^t d\tau\right]^2 \leq \int_0^t \frac{d\tau}{X_\tau} \int_0^t X_\tau\, d\tau,$$

and thus

$$p\int_0^t \frac{d\tau}{X_\tau} \geq \frac{pt^2}{\displaystyle\int_0^t X_\tau\, d\tau}.$$

The result follows from (b). □

By contrast, it is easy to see that the density of X_t behaves like $x^{\bar{v}-1}$ as $x \downarrow 0$, with $\bar{v} = 2\beta/\gamma^2$ (see (2.17) and Theorem 2.4), and thus $\mathsf{E}X_t^p < \infty$ for $p > -\bar{v}$ only, whatever the initial condition $\bar{x} \geq 0$, and, consequently, $\mathsf{E}\exp(q/X_t) = \infty$ for any $q > 0$. Thus, the density of Y_t tends to 0 much quicker than that of X_t, as the argument tends to 0. This is a little unexpected, especially since $Y_0 = 0$ with probability one, irrespective of $X_0 = \bar{x}$. An intuitive explanation is that Y_t is close to 0 if X_s is close to 0 for "many" s between 0 and t, which is less likely than X_s being close to 0 for one particular s; integration is therefore a "smoothing" operation in this case. This relates to Theorem 4.2 below, which says that the density of Y_t is "infinitely flat" at the origin (that is, all its derivatives vanish).

Part (c) of the theorem is related to absence of arbitrage and changes of measures in the financial model

$$dS \;=\; \mu S\,dt + \sqrt{X}S\,dV,$$

where X satisfies (1.1) and (V, W) is two-dimensional Brownian motion (possibly correlated). If one attempts the usual Girsanov change of measure to make $\{e^{-rt}S_t\}$ a martingale, then one has the problem of checking that the stochastic exponential of minus the integral of the market price of risk

$$\frac{\mu - r}{\sqrt{X}}$$

has expectation equal to one (which would make it a martingale, see Karatzas & Shreve (1998), p.12). A sufficient condition for this to hold is Novikov's condition

$$\mathsf{E}\exp\left(\frac{1}{2}\int_0^t \frac{(\mu-r)^2}{X_s}\,ds\right) \;<\; \infty.$$

Theorem 4.1 shows that Novikov's condition fails if

$$(\mu - r)^2 \;\geq\; \frac{1}{\gamma^2}\left(\beta + \frac{\bar{x}}{t}\right)^2$$

(but does not say that the condition holds otherwise).

Observe that, by Theorem 2.4, $\mathsf{E}(1/X_s) = \infty$ for all $s > 0$ if $2\beta/\gamma^2 \leq 1$, which implies

$$\mathsf{E}\int_0^t \frac{1}{X_s}ds \;=\; \infty,$$

and thus $\mathsf{E}e^{p\int_0^t ds/X_s} = \infty$ for all $p > 0$. More generally, since $\mathsf{E}X_t^{-q} = \infty$ for all $q \geq 2\beta/\gamma^2$, it is plausible that not all moments of $\int_0^t \frac{1}{X_s}ds$ are finite (though the author has not been able to prove this), which would then imply $\mathsf{E}e^{p\int_0^t ds/X_s} = \infty$ for all $p > 0$, for any β, γ and $t > 0$.

Theorem 4.2. *Suppose $t > 0$ and let $g_t(\cdot)$ be the density of Y_t with respect to Lebesgue measure on $\mathbb{R}$. Then $g_t(x)$ exists and is a continuous, uniformly bounded function of $x \in \mathbb{R}$. The same applies to $(d^n/dx^n)g_t(x)$,* for any $n \geq 1$.

Proof. Apply the following well-known property of characteristic functions (see Theorem 3 and Lemma 2 in Feller (1971), pp.509 and 512): *Suppose a r.v. U satisfies*

$$\int_{-\infty}^{\infty} |\mathsf{E}\, e^{i\zeta U}\zeta^n|\, d\zeta \; < \; \infty$$

for some $n \geq 1$; then the distribution of U has a density which is everywhere continuous and uniformly bounded, and the nth derivative of this density is also everywhere continuous and uniformly bounded.

The characteristic function of Y_t is

$$\begin{aligned}\psi(\zeta) &= \mathsf{E}\, e^{i\zeta Y_t} = \mathsf{E}\, e^{-sY_t}\Big|_{s=-i\zeta} \\ &= \left[\frac{e^{-\alpha t/2}}{\cosh(Zt/2) - \frac{\alpha}{Z}\sinh(Zt/2)}\right]^{\frac{2\beta}{\gamma^2}} \exp\left[\frac{i\zeta\bar{x}}{Z}\frac{2\sinh(Zt/2)}{\cosh(Zt/2) - \frac{\alpha}{Z}\sinh(Zt/2)}\right],\end{aligned}$$

where $Z = Z(\zeta) = \sqrt{\alpha^2 - 2\gamma^2\zeta i}$. First, consider $\psi(\zeta)$ as $\zeta \to \infty$. We have (see (2.14))

$$\begin{aligned}Z(\zeta) - \gamma(1-i)\sqrt{\zeta} &= \gamma(1-i)\sqrt{\zeta}\left(\sqrt{1 - \frac{\alpha^2}{2\gamma^2\zeta i}} - 1\right) \\ &= \gamma(1-i)\sqrt{\zeta}\sum_{k=1}^{\infty}\left(\frac{\alpha^2}{2\gamma^2\zeta i}\right)^k \frac{(-\frac{1}{2})_k}{k!} \\ &\to 0\end{aligned}$$

as $\zeta \to \infty$. This implies

$$\begin{aligned}e^{Zt/2} &\sim e^{\gamma(1-i)\sqrt{\zeta}t/2} \\ \cosh(Zt/2) - \frac{\alpha}{Z}\sinh(Zt/2) &\sim \frac{1}{2}e^{\gamma(1-i)\sqrt{\zeta}t/2}.\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{i\zeta\bar{x}}{Z}\frac{2\sinh(Zt/2)}{\cosh(Zt/2) - \frac{\alpha}{Z}\sinh(Zt/2)} &= \frac{2i\zeta\bar{x}}{Z}\frac{1}{\coth(Zt/2) - \frac{\alpha}{Z}} \\ &= \frac{(-1+i)\bar{x}\sqrt{\zeta}}{\gamma}\left(1 - \frac{\alpha^2}{2\gamma^2\zeta i}\right)^{-\frac{1}{2}} \frac{1}{\frac{1+e^{-Zt}}{1-e^{-Zt}} - \frac{\alpha}{\gamma(1-i)\sqrt{\zeta}}\left(1 - \frac{\alpha^2}{2\gamma^2\zeta i}\right)^{-\frac{1}{2}}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1+i)\bar{x}\sqrt{\zeta}}{\gamma}\left(1+\mathcal{O}(\zeta^{-1})\right)\frac{1}{1+\mathcal{O}(e^{-Zt})-\frac{\alpha}{\gamma(1-i)\sqrt{\zeta}}\left(1+\mathcal{O}(\zeta^{-1})\right)} \\
&= \frac{(-1+i)\bar{x}\sqrt{\zeta}}{\gamma}\left(1+\mathcal{O}(\zeta^{-1})\right)\left(1+\frac{\alpha}{\gamma(1-i)\sqrt{\zeta}}+\mathcal{O}(\zeta^{-1})\right) \\
&= \frac{(-1+i)\bar{x}\sqrt{\zeta}}{\gamma}-\frac{\alpha\bar{x}}{\gamma^2}+\mathcal{O}(\zeta^{-\frac{1}{2}})
\end{aligned}
$$

Putting all this together, we find

$$|\psi(\zeta)| \;\sim\; C_1 e^{-C_2\sqrt{\zeta}}$$

as $\zeta \to \infty$, where $C_1, C_2 > 0$ do not depend on ζ. Hence

$$\int_0^\infty |\psi(\zeta)\zeta^n|\,d\zeta \;<\; \infty$$

for any $n \geq 0$. The integral from $-\infty$ to 0 is handled similarly: as $\zeta \to -\infty$,

$$
\begin{aligned}
Z(\zeta)-\gamma(1+i)\sqrt{-\zeta} &\;\to\; 0, \\
e^{Zt/2} &\;\sim\; e^{\gamma(1+i)\sqrt{-\zeta}t/2} \\
\cosh(Zt/2)-\frac{\alpha}{Z}\sinh(Zt/2) &\;\sim\; \frac{1}{2}e^{\gamma(1+i)\sqrt{-\zeta}t/2} \\
\frac{i\zeta\bar{x}}{Z}\frac{2\sinh(Zt/2)}{\cosh(Zt/2)-\frac{\alpha}{Z}\sinh(Zt/2)} &\;=\; -\frac{(1+i)\bar{x}\sqrt{-\zeta}}{\gamma}-\frac{\alpha\bar{x}}{\gamma^2}+\mathcal{O}((-\zeta)^{-\frac{1}{2}}) \\
|\psi(\zeta)| &\;\sim\; C_3 e^{-C_4\sqrt{-\zeta}}.
\end{aligned}
$$

□

5. Relationships between the Laplace transforms of U and $1/U$

The formula in Theorem 4.1 (b) applies for any random variable $U > 0$. The same proof leads to the identity

$$\mathsf{E}\,e^{p/U} \;=\; 1+\int_0^\infty \sqrt{\frac{p}{s}}\,I_1(2\sqrt{ps})\,\mathsf{E}\exp(-sU)\,ds, \qquad p \geq 0 \tag{5.1}$$

(both sides are simultaneously finite or infinite, by Fubini's Theorem). Formula (5.1) strangely resembles a known relationship for Laplace transforms (Oberhettinger & Badii, 1973, p.4):

$$\int_0^\infty e^{-pt}t^{\nu-1}f(t^{-1})\,dt \;=\; \int_0^\infty \left(\frac{s}{p}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{ps})\left[\int_0^\infty e^{-st}f(t)\,dt\right]ds, \qquad \nu > -1. \tag{5.2}$$

Here, J_ν is the ordinary Bessel function of order ν (Lebedev, 1972, p.102),

$$J_\nu(z) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \qquad |\arg z| < \pi, \qquad \nu \in \mathbb{C}.$$

Let us rephrase (5.2) in probabilistic terms, by considering that $f(\cdot)$ is the density of a random variable $U > 0$:

$$\mathsf{E}\,U^{-\nu-1} e^{-p/U} \;=\; \int_0^\infty \left(\frac{s}{p}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{ps})\,\mathsf{E}\,e^{-sU}\,ds, \qquad \nu > -1. \tag{5.3}$$

(Oberhettinger & Badii do not specifiy for which p this holds, but this is clarified in the theorem below.) Observe that $\nu = -1$ is not allowed here, so that (5.3) does not apply to $\mathsf{E}\,e^{-p/U}$. We now derive some more general formulas, which have (5.1) and (5.3) as particular cases.

Theorem 5.1. *Suppose $p > 0$, $r \in \mathbb{R}$ and $U > 0$ a.s.*

(a) If $\nu > -1$, then

$$\mathsf{E}\,U^{r-\nu-1} e^{p/U} \;=\; \int_0^\infty \left(\frac{s}{p}\right)^{\frac{\nu}{2}} I_\nu(2\sqrt{ps})\,\mathsf{E}\,U^r e^{-sU}\,ds.$$

Both sides may be infinite.

(b) If $\nu \le -1$, then

$$\begin{aligned}\mathsf{E}\,U^{r-\nu-1} e^{p/U} \;=\; & \sum_{0\le n\le -\nu-1} \frac{p^n}{n!}\,\mathsf{E}\,U^{r-\nu-n-1} \\ & + \int_0^\infty \left[\left(\frac{s}{p}\right)^{\frac{\nu}{2}} I_\nu(2\sqrt{ps}) - \sum_{0\le n\le -\nu-1} \frac{p^n s^{n+\nu}}{n!\Gamma(n+\nu+1)}\right] \mathsf{E}\,U^r e^{-sU}\,ds.\end{aligned}$$

Both sides may be infinite. If, moreover, $-\nu - 1 \in \mathbb{N}$, then

$$\mathsf{E}\,U^{r-\nu-1} e^{p/U} \;=\; \sum_{0\le n\le -\nu-1} \frac{p^n}{n!}\,\mathsf{E}\,U^{r-\nu-n-1} + \int_0^\infty \left(\frac{s}{p}\right)^{\frac{\nu}{2}} I_{-\nu}(2\sqrt{ps})\,\mathsf{E}\,U^r e^{-sU}\,ds.$$

(c) If $\nu > -1$, $\mathsf{E}\,U^r < \infty$, $\mathsf{E}\,U^{r-\nu-1} < \infty$, then

$$\mathsf{E}\,U^{r-\nu-1} e^{-p/U} \;=\; \int_0^\infty \left(\frac{s}{p}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{ps})\,\mathsf{E}\,U^r e^{-sU}\,ds.$$

Both sides are finite, and the integral on the right may be improper (that is, it may not converge absolutely) if $\nu < -1/2$.

(d) If $\nu \leq -1$, $\mathsf{E}U^r < \infty$, $\mathsf{E}U^{r-\nu-1} < \infty$, then

$$\begin{aligned}\mathsf{E}U^{r-\nu-1}e^{-p/U} &= \sum_{0\leq n\leq -\nu-1} \frac{(-p)^n}{n!}\mathsf{E}U^{r-\nu-n-1} \\ &\quad + \int_0^\infty \left[\left(\frac{s}{p}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{ps}) - \sum_{0\leq n\leq -\nu-1}\frac{(-p)^n s^{n+\nu}}{n!\Gamma(n+\nu+1)}\right]\mathsf{E}U^r e^{-sU}\,ds.\end{aligned}$$

Both sides are finite, and the integral on the right may be improper. If, moreover, $-\nu-1 \in \mathbb{N}$, then

$$\mathsf{E}U^{r-\nu-1}e^{-p/U} = \sum_{0\leq n\leq -\nu-1} \frac{(-p)^n}{n!}\mathsf{E}U^{r-\nu-n-1} + (-1)^\nu \int_0^\infty \left(\frac{s}{p}\right)^{\frac{\nu}{2}} J_{-\nu}(2\sqrt{ps})\,\mathsf{E}U^r e^{-sU}\,ds.$$

Proof. (a) Because all the terms in the following series are positive, the equalities hold whether the resulting expressions are finite or infinite:

$$\begin{aligned}\mathsf{E}U^{r-\nu-1}\,e^{p/U} &= \sum_{n=0}^\infty \frac{p^n}{n!}\mathsf{E}U^{r-\nu-n-1} \\ &= \sum_{n=0}^\infty \frac{p^n}{n!}\mathsf{E}U^r\int_0^\infty \frac{s^{n+\nu}}{\Gamma(\nu+n+1)}e^{-sU}\,ds \\ &= \int_0^\infty\left(\sum_{n=0}^\infty \frac{p^n}{n!}\,\frac{s^{n+\nu}}{\Gamma(\nu+n+1)}\right)\mathsf{E}U^r e^{-sU}\,ds \\ &= \int_0^\infty \left(\frac{s}{p}\right)^{\frac{\nu}{2}} I_\nu(2\sqrt{ps})\,\mathsf{E}U^r e^{-sU}\,ds.\end{aligned}$$

(b) The first formula results from

$$\mathsf{E}U^{r-\nu-1}\,e^{p/U} = \sum_{0\leq n\leq -\nu-1}\frac{p^n}{n!}\mathsf{E}U^{r-\nu-n-1} + \int_0^\infty \sum_{n>-\nu-1}\frac{p^n s^{n+\nu}}{n!\Gamma(\nu+n+1)}\mathsf{E}U^r\,e^{-sU}\,ds.$$

The second one then follows from (Lebedev, 1972, p.110)

$$\frac{1}{\Gamma(-m)} = 0, \qquad I_{-m}(z) = I_m(z), \quad m\in\mathbb{N}.$$

(c) Let $U_\epsilon = \max(U,\epsilon)$ for $\epsilon \geq 0$ (whence $U_0 = U$). For $\epsilon > 0$, $1/U_\epsilon$ is bounded below by 0, and above by $1/\epsilon$, whence

$$\mathsf{E}U^r U_\epsilon^{-\nu-1}\,e^{p/U_\epsilon} < \infty.$$

Therefore the series below converge absolutely, and so it is permitted to reverse the order of summation, integration and expectation at will:

$$\begin{aligned}
\mathsf{E}U^r U_\epsilon^{-\nu-1} e^{-p/U_\epsilon} &= \int_0^\infty \left(\sum_{n=0}^\infty \frac{(-p)^n}{n!} \frac{s^{n+\nu}}{\Gamma(\nu+n+1)} \right) \mathsf{E}U^r e^{-sU_\epsilon}\, ds \\
&= \int_0^\infty \left(\frac{s}{p}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{ps})\, \mathsf{E}U^r e^{-sU_\epsilon}\, ds. \qquad (5.4)
\end{aligned}$$

Now $-\nu - 1 < 0$, so

$$0 \le U^r U_\epsilon^{-\nu-1} e^{-p/U_\epsilon} \le CU^r$$

where $C = \sup_{x>0} x^{\nu+1} e^{-px} < \infty$, and U^r is integrable by assumption. Thus

$$\mathsf{E}U^r U_\epsilon^{-\nu-1} e^{-p/U_\epsilon} \to \mathsf{E}U^{r-\nu-1} e^{-p/U}$$

as $\epsilon \to 0+$. Next, turn to the right hand side of (5.4).

Case 1: $\nu \ge -\frac{1}{2}$

The asymptotic expression (Lebedev, 1972, p.122)

$$J_\nu(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - (2\nu+1)\tfrac{\pi}{4}\right) + \mathcal{O}(x^{-3/2}) \qquad \text{as } x \to \infty$$

implies that there is $C > 0$ such that

$$|J_\nu(x)| \le Cx^{-\frac{1}{2}}, \qquad x > 0.$$

Therefore

$$\left| \left(\frac{s}{p}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{ps})\, \mathsf{E}U^r e^{-sU_\epsilon} \right| \le C_1 s^{\frac{\nu}{2}-\frac{1}{4}}\, \mathsf{E}U^r e^{-sU}.$$

Now $\nu \ge -\frac{1}{2}$ implies $\frac{\nu}{2} + \frac{3}{4} > 0$, and so

$$\int_0^\infty s^{\frac{\nu}{2}-\frac{1}{4}}\, \mathsf{E}U^r e^{-sU}\, ds = \mathsf{E}U^{r-\frac{\nu}{2}-\frac{3}{4}}\, \Gamma\left(\tfrac{\nu}{2}+\tfrac{3}{4}\right).$$

This is finite, because $r - \nu - 1 \le r - \frac{\nu}{2} - \frac{3}{4} \le r$, and it is assumed that both $\mathsf{E}U^r$ and $\mathsf{E}U^{r-\nu-1}$ are finite. Finally, the right hand side of (5.4) converges to

$$\int_0^\infty \left(\frac{s}{p}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{ps})\, \mathsf{E}U^r e^{-sU}\, ds$$

as $\epsilon \to 0+$, by dominated convergence.

Case 2: $\nu \in (-1, -\frac{1}{2})$ The above arguments break down if $\nu < -\frac{1}{2}$, because then

$$r - \frac{\nu}{2} - \frac{3}{4} < r - \nu - 1.$$

The problem can be solved either by assuming that $\mathsf{E}U^{r-\frac{\nu}{2}-\frac{3}{4}} < \infty$ or else as follows.

Write the integral on the right hand side of (5.4) as $\int_0^1 + \int_1^\infty$. It is immediately seen that

$$\int_0^1 \left(\frac{s}{p}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{ps})\,\mathsf{E}U^r e^{-sU_\epsilon}\,ds \;\to\; \int_0^1 \left(\frac{s}{p}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{ps})\,\mathsf{E}U^r e^{-sU}\,ds$$

as $\epsilon \to 0+$, by dominated convergence. For the second integral, consider separately each term on the right of

$$J_\nu(2\sqrt{ps}) \;=\; C_2 s^{-\frac{1}{4}} \cos\left(2\sqrt{ps} - (2\nu+1)\tfrac{\pi}{4}\right) + R(s) \qquad \text{as } s \to \infty$$

where $C_2 > 0$ and $R(s) = \mathcal{O}(s^{-\frac{3}{4}})$. Since $\frac{\nu}{2} - \frac{3}{4} < -1$, the integral

$$\int_1^\infty \left(\frac{s}{p}\right)^{\frac{\nu}{2}} R(s)\,\mathsf{E}U^r e^{-sU_\epsilon}\,ds$$

converges absolutely for $\epsilon \geq 0$, and so convergence as $\epsilon \to 0+$ to

$$\int_1^\infty \left(\frac{s}{p}\right)^{\frac{\nu}{2}} R(s)\,\mathsf{E}U^r e^{-sU}\,ds$$

follows at once. The only remaining problem is to show that

$$\int_0^\infty s^{\frac{\nu}{2}-\frac{1}{4}} \cos\left(2\sqrt{ps} - (2\nu+1)\tfrac{\pi}{4}\right)\,\mathsf{E}U^r e^{-sU_\epsilon}\,ds$$

converges to the same expression with $\epsilon = 0$, as $\epsilon \to 0+$. Perform the change of variable $u = 2\sqrt{ps}$, $s = u^2/(4p)$, to obtain

$$\begin{aligned}
2\int_{2\sqrt{p}}^\infty &\left(\frac{u^2}{4p}\right)^{\frac{\nu}{2}-\frac{1}{4}} u\cos(u-(2\nu+1)\tfrac{\pi}{4})\,\mathsf{E}U^r e^{-u^2U_\epsilon/(4p)}\,du \\
&= C_3\int_{2\sqrt{p}}^\infty u^{\nu+\frac{1}{2}}\cos(u-(2\nu+1)\tfrac{\pi}{4})\,\mathsf{E}U^r e^{-u^2U_\epsilon/(4p)}\,du \\
&= C_3\left\{u^{\nu+\frac{1}{2}}\sin(u-(2\nu+1)\tfrac{\pi}{4})\,\mathsf{E}U^r e^{-u^2U_\epsilon/(4p)}\Big|_{u=2\sqrt{p}}^{u\to\infty}\right\} \\
&\quad - C_3\int_{2\sqrt{p}}^\infty \sin(u-(2\nu+1)\tfrac{\pi}{4})\,d\left[u^{\nu+\frac{1}{2}}\,\mathsf{E}U^r e^{-u^2U_\epsilon/(4p)}\right].
\end{aligned} \tag{5.5}$$

Now $u^{\nu+\frac{1}{2}}$ tends to 0 as u tends to ∞, so the expression inside the curly brackets reduces to

$$-u^{\nu+\frac{1}{2}}\sin(u-(2\nu+1)\tfrac{\pi}{4})\,\mathsf{E}U^r e^{-u^2U_\epsilon/(4p)}\Big|_{u=2\sqrt{p}},$$

which poses no problem. The integral in (5.5) converges absolutely, since

$$\lim_{u\to\infty} u^{\nu+\frac{1}{2}}\,\mathsf{E}U^r e^{-u^2U_\epsilon/(4p)} \;=\; 0.$$

To prove that it converges to

$$\int_{2\sqrt{p}}^{\infty}\sin(u-(2\nu+1)\tfrac{\pi}{4})\,d\left[u^{\nu+\frac{1}{2}}\,\mathsf{E}U^r e^{-u^2U/(4p)}\right],$$

apply the Helly-Bray Theorem (Loève, 1977, p.184; Kolmogorov & Fomine, 1975, p.370). The functions

$$F_\epsilon(u)=u^{\nu+\frac{1}{2}}\,\mathsf{E}U^r e^{-u^2U_\epsilon/(4p)},\qquad \epsilon\ge 0$$

are of bounded variation on $\mathbb{R}_+$, and $\{F_\epsilon;\epsilon>0\}$ converge completely (Loève, 1977, p.180) to F_0 as $\epsilon\to 0+$. Since $g(u)=\sin(u-(2\nu+1)\frac{\pi}{4})$ is bounded, we conclude that

$$\int_{2\sqrt{p}}^{\infty} g\,dF_\epsilon \;\to\; \int_{2\sqrt{p}}^{\infty} g\,dF_0 \qquad \text{as}\quad \epsilon\to 0+.$$

(d) Proceed as in (b) and (c) to get

$$\begin{aligned}
&\mathsf{E}U^r U_\epsilon^{-\nu-1}\,e^{p/U_\epsilon}\\
&= \sum_{0\le n\le -\nu-1}\frac{(-p)^n}{n!}\,\mathsf{E}U^rU_\epsilon^{-\nu-n-1}\\
&\quad+\int_0^\infty\left[\left(\frac{s}{p}\right)^{\frac{\nu}{2}}J_\nu(2\sqrt{ps})-\sum_{0\le n\le -\nu-1}\frac{(-p)^n s^{n+\nu}}{n!\Gamma(\nu+n+1)}\right]\mathsf{E}U^r\,e^{-sU_\epsilon}\,ds
\end{aligned}$$

for any $\epsilon>0$. The first sum poses no problem. Split the integral into $\int_0^1+\int_1^\infty$. The former tends to the right limit by dominated convergence. To prove convergence of $\int_1^\infty$, consider each term of the integrand separately. The first one converges to

$$\int_1^\infty\left(\frac{s}{p}\right)^{\frac{\nu}{2}}J_\nu(2\sqrt{ps})\,\mathsf{E}U^r\,e^{-sU}\,ds$$

by the arguments given in Case 2 of (c). Each of the other terms gives rise to an absolutely convergent integral, since $n+\nu<-1$, and dominated convergence does the rest.

Finally, in the special case $-\nu-1\in\mathbb{N}$, the sum inside the integral vanishes (as in (b)), and one applies (Lebedev, 1972, p.103)

$$J_{-n}(z) \;=\; (-1)^n J_n(z),\qquad n=1,2,\ldots \qquad \square$$

Theorem 4.1(b) results from letting $\nu = -1$ in Theorem 5.1(b). This theorem has the following consequences, which relate the existence of $\mathsf{E}U^{-\nu-1}e^{p/U}$, which is itself determined by the behaviour of the distribution of U near the origin, to the behaviour of $\mathsf{E}e^{-sU}$ near infinity.

Corollary 5.2. *Suppose $p > 0$, $\nu \in \mathbb{R}$ and $U > 0$ a.s.*

(a) If $\mathsf{E}U^{-\nu-1}e^{p/U} < \infty$, then

$$\liminf_{s\to\infty} s^{\frac{\nu}{2}-\frac{1}{4}}\, e^{2\sqrt{ps}}\,\mathsf{E}e^{-sU} \;=\; 0.$$

(b) If $\nu > -1$ and if, for some $\epsilon > 0$, $\mathsf{E}e^{-sU} = \mathcal{O}(s^{-\frac{3}{4}-\frac{\nu}{2}-\epsilon}e^{-2\sqrt{ps}})$ as $s \to \infty$, then $\mathsf{E}U^{-\nu-1}e^{p/U} < \infty$. The same holds for $\nu \leq -1$, with the additional assumption $\mathsf{E}U^{-\nu-1} < \infty$.

Proof. (a) First suppose $\nu > -1$, and let $r = 0$ in Theorem 5.1(a). If the integral converges, then necessarily the integrand must have a limit inferior equal to 0. By (4.3),

$$\liminf_{s\to\infty} s^{\frac{\nu}{2}} I_\nu(2\sqrt{ps})\,\mathsf{E}e^{-sU} \;=\; C \liminf_{s\to\infty} s^{\frac{\nu}{2}-\frac{1}{4}}\, e^{2\sqrt{ps}}\,\mathsf{E}e^{-sU},$$

where $C > 0$ is a constant. For $\nu \leq -1$, let $r = 0$ in Theorem 5.1(b). If the left hand side of the first formula is finite, then the integral on the other side must be finite. The situation is seen to be the same as when $\nu > -1$, once it is observed that

$$\int_1^\infty s^{n+\nu}\,\mathsf{E}e^{-sU}\,ds \;<\; \infty$$

for any $n < \nu + 1$.

(b) Suppose $\nu > -1$. The assumptions imply that there is a constant C such that

$$\begin{aligned}
\mathsf{E}U^{-\nu-1}e^{p/U} \;&=\; \left[\int_0^1 + \int_1^\infty\right] \left(\frac{s}{p}\right)^{\frac{\nu}{2}} I_\nu(2\sqrt{ps})\,\mathsf{E}U^r e^{-sU}\,ds \\
&\leq\; \int_0^1 \left(\frac{s}{p}\right)^{\frac{\nu}{2}} I_\nu(2\sqrt{ps})\,\mathsf{E}U^r e^{-sU}\,ds + C\int_1^\infty s^{\frac{\nu}{2}} I_\nu(2\sqrt{ps})\, s^{-\frac{3}{4}-\frac{\nu}{2}-\epsilon}\, e^{-2\sqrt{ps}}\,ds \\
&=\; \int_0^1 \left(\frac{s}{p}\right)^{\frac{\nu}{2}} I_\nu(2\sqrt{ps})\,\mathsf{E}U^r e^{-sU}\,ds + C\int_1^\infty s^{\frac{1}{4}}\, e^{-2\sqrt{ps}} I_\nu(2\sqrt{ps})\, s^{-1-\epsilon}\,ds
\end{aligned}$$

The result follows from (4.3). If $\nu \leq -1$ and, moreover, $\mathsf{E}U^{-\nu-1} < \infty$, then

$$\mathsf{E}U^{-\nu-n-1} < \infty \qquad \forall\; 0 \leq n \leq -\nu-1,$$

$$\int_1^\infty s^{n+\nu}\,\mathsf{E}e^{-sU}\,ds \;<\; \infty \qquad \forall\; 0 \leq n < -\nu-1,$$

and

$$\int_1^\infty s^{\frac{\nu}{2}} I_\nu(2\sqrt{ps}) \,\mathsf{E}\, e^{-sU}\, ds \;<\; \infty.$$

This implies $\mathsf{E}U^{-\nu-1}e^{p/U} < \infty$, by Theorem 5.1(b). $\square$

Example. Macdonald's function is defined in terms of the modified Bessel function as follows (Lebedev, 1972):

$$\begin{aligned} K_\nu(z) &= \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \quad |\arg z| < \pi,\ \nu \notin \mathbb{Z}, \\ K_n(z) &= \lim_{\nu\to n} K_\nu(z), \quad n \in \mathbb{Z}. \end{aligned}$$

It also has the integral expression

$$K_\nu(z) \;=\; \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t-(z^2/4t)} t^{-\nu-1}\, dt, \quad |\arg(z)| < \frac{\pi}{4}.$$

Clearly $K_{-\nu}(z) = K_\nu(z)$. The generalized inverse Gaussian distribution with parameters $\alpha > 0, \beta > 0, \gamma \in \mathbb{R}$ has density

$$f(x) \;=\; \left(\frac{\alpha}{\beta}\right)^{\frac{\gamma}{2}} \frac{1}{2K_\gamma(\sqrt{\alpha\beta})} x^{\gamma-1} \exp\left\{-\frac{1}{2}\left(\alpha x + \frac{\beta}{x}\right)\right\} \mathbf{1}_{(0,\infty)}(x).$$

The notation for this distribution is $\mathbf{GIG}(\alpha,\beta,\gamma)$. It is immediate that $U \sim \mathbf{GIG}(\alpha,\beta,\gamma)$ implies $(1/U) \sim \mathbf{GIG}(\beta,\alpha,-\gamma)$. Moreover, a simple calculation shows that

$$\mathsf{E}\, U^r e^{-sU} \;=\; \frac{\alpha^{\frac{\gamma}{2}}\beta^{\frac{r}{2}}}{(\alpha+2s)^{\frac{\gamma+r}{2}}} \frac{K_{\gamma+r}(\sqrt{(\alpha+2s)\beta})}{K_\gamma(\sqrt{\alpha\beta})}, \quad \mathrm{Re}(s) > -\frac{\alpha}{2}.$$

Theorem 5.1(c) then says that

$$\begin{aligned} &\int_0^\infty \left(\frac{s}{p}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{ps}) \frac{\alpha^{\frac{\gamma}{2}}\beta^{\frac{r}{2}}}{(\alpha+2s)^{\frac{\gamma+r}{2}}} \frac{K_{\gamma+r}(\sqrt{(\alpha+2s)\beta})}{K_\gamma(\sqrt{\alpha\beta})}\, ds \\ &\qquad = \frac{\beta^{-\frac{\gamma}{2}}\alpha^{\frac{1+\nu-r}{2}}}{(\beta+2p)^{\frac{-\gamma+1+\nu-r}{2}}} \frac{K_{1+\nu-r-\gamma}(\sqrt{(\beta+2p)\alpha})}{K_\gamma(\sqrt{\alpha\beta})} \end{aligned}$$

or, equivalently,

$$\begin{aligned} &\int_0^\infty \left(\frac{s}{p}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{ps}) \frac{K_{\gamma+r}(\sqrt{(\alpha+2s)\beta})}{(\alpha+2s)^{\frac{\gamma+r}{2}}}\, ds \\ &\qquad = \frac{\beta^{-\frac{\gamma+r}{2}}\alpha^{\frac{1+\nu-r-\gamma}{2}}}{(\beta+2p)^{\frac{-\gamma+1+\nu-r}{2}}} K_{1+\nu-r-\gamma}(\sqrt{(\beta+2p)\alpha}). \quad (5.6) \end{aligned}$$

By performing the change of variable $x = 2\sqrt{ps}/b$, and setting

$$\alpha = \frac{b^2y^2}{2p}, \quad \beta = \frac{2a^2p}{b^2}, \quad \mu = \gamma + r,$$

it can be seen that (5.6) is equivalent to the formula (Lebedev, 1972, p. 134)

$$\int_0^\infty \frac{K_\mu(a\sqrt{x^2+y^2})}{(x^2+y^2)^{\mu/2}} J_\nu(bx) x^{\nu+1}\,dx \;=\; \frac{b^\nu}{a^\mu}\left(\frac{\sqrt{a^2+b^2}}{y}\right)^{\mu-\nu-1} K_{\mu-\nu-1}(y\sqrt{a^2+b^2}). \tag{5.7}$$

An extension of this result is obtained by applying part (d) of Theorem 5.1 to the same distribution $\mathbf{GIG}(\alpha, \beta, \gamma)$. With the same change of variable and substitutions, we get

$$\begin{aligned}\int_0^\infty & \frac{K_\mu(a\sqrt{x^2+y^2})}{(x^2+y^2)^{\mu/2}} \left[J_\nu(bx) - \sum_{0\le n<-\nu-1} \frac{(-1)^n(bx/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)}\right] x^{\nu+1}\,dx \\ = \;& \frac{b^\nu}{a^\mu}\left(\frac{\sqrt{a^2+b^2}}{y}\right)^{\mu-\nu-1} K_{\mu-\nu-1}(y\sqrt{a^2+b^2}) - \frac{b^\nu}{a^{\nu+1}y^{\mu-\nu-1}} \sum_{0\le n\le -\nu-1} \frac{1}{n!}\left(-\frac{b^2y}{2a}\right)^n K_{\mu-\nu-n-1}(ay).\end{aligned} \tag{5.8}$$

The Appendix shows another way of deriving the same formula.

6. Conclusion

This paper has shown how the moments of the integral of the square-root process may be computed. Of the three methods shown, the the recursive method (Theorem 3.2) is the fastest to execute, and its programming poses no problem. The paper also showed some regularly properties of the density of the integral.

Further work will consider expressing the density of Y_t as an infinite Laguerre series, as was done for the density of the integral of geometric Brownian motion in Dufresne (2000). The coefficients of the Laguerre polynomials are combinations of moments of the variable, and this is where the recursive equations of Section 3 will be essential, as they generate the moments much faster than differentiating the Laplace transform. A finite number of terms of the series would then serve as an approximation for the density. The same approach may be used for options on integrated squared volatility, or for options on average interest rates. Alternative methods are inversion of the Laplace transform, and Monte Carlo simulation.

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APPENDIX

We give another proof of formula (5.8). First, we derive a slight extension of Weber's integral (Lebedev, 1972, p.132)

$$\int_0^\infty e^{-a^2x^2} J_\nu(bx) x^{\nu+1}\,dx \;=\; \frac{b^\nu}{(2a^2)^{\nu+1}} e^{-b^2/4a^2}, \qquad a,b>0, \quad \mathrm{Re}(\nu) > -1.$$

Leave $a, b > 0$, but now suppose $\nu \leq -1$ (the case of complex ν with $\mathrm{Re}(\nu) \leq -1$ remains to be explored). Consider

$$\begin{aligned}
&\int_0^\infty e^{-a^2x^2}\left[J_\nu(bx) - \sum_{0\leq n\leq -\nu-1} \frac{(-1)^n (bx/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}\right] x^{\nu+1}\,dx \\
&\qquad = \sum_{n>-\nu-1} \frac{(-1)^n (b/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)} \int_0^\infty e^{-a^2x^2} x^{2\nu+2n+1}\,dx \\
&\qquad = \sum_{n>-\nu-1} \frac{(-1)^n (b/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)} \frac{1}{2} a^{-2\nu-2n-2} \int_0^\infty e^{-u} u^{\nu+n}\,du \\
&\qquad = \sum_{n>-\nu-1} \frac{(-1)^n (b/2)^{\nu+2n}}{n!} \frac{1}{2} a^{-2\nu-2n-2} \\
&\qquad = \frac{b^\nu}{(2a^2)^{\nu+1}}\left[e^{-b^2/4a^2} - \sum_{0\leq n\leq -\nu-1} \frac{1}{n!}\left(-\frac{b^2}{4a^2}\right)^n\right].
\end{aligned}$$

From this formula, we obtain

$$\begin{aligned}
&\int_0^\infty \frac{K_\mu(a\sqrt{x^2+y^2})}{(x^2+y^2)^{\mu/2}}\left[J_\nu(bx) - \sum_{0\leq n<-\nu-1} \frac{(-1)^n (bx/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)}\right] x^{\nu+1}\,dx \\
&= \int_0^\infty dx \left[J_\nu(bx) - \sum_{0\leq n\leq -\nu-1} \frac{(-1)^n (bx/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)}\right] x^{\nu+1} \frac{a^\mu}{2^{\mu+1}} \int_0^\infty \frac{dt}{t^{\mu+1}} e^{-t-a^2(x^2+y^2)/4t} \\
&= \frac{a^\mu}{2^{\mu+1}} \int_0^\infty \frac{dt}{t^{\mu+1}} e^{-t-a^2y^2/4t} \int_0^\infty dx\, e^{-a^2x^2/4t}\left[J_\nu(bx) - \sum_{0\leq n\leq -\nu-1} \frac{(-1)^n (bx/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)}\right] x^{\nu+1} \\
&= \frac{a^\mu}{2^{\mu+1}} \int_0^\infty \frac{dt}{t^{\mu+1}} e^{-t-a^2y^2/4t} \frac{b^\nu}{(a^2/2t)^{\nu+1}}\left[e^{-b^2t/a^2} - \sum_{0\leq n\leq -\nu-1} \frac{1}{n!}\left(-\frac{b^2t}{a^2}\right)^n\right] \\
&= 2^{\nu-\mu} a^{\mu-2\nu-2} b^\nu \left[\int_0^\infty \frac{dt}{t^{\mu-\nu}} e^{-t-a^2y^2/4t-b^2t/a^2} - \sum_{0\leq n\leq -\nu-1} \frac{1}{n!}\left(-\frac{b^2}{a^2}\right)^n \int_0^\infty \frac{dt}{t^{\mu-\nu-n}} e^{-t-a^2y^2/4t}\right. \\
&= \frac{b^\nu}{a^\mu}\left(\frac{\sqrt{a^2+b^2}}{y}\right)^{\mu-\nu-1} K_{\mu-\nu-1}(y\sqrt{a^2+b^2}) - \frac{b^\nu}{a^{\nu+1} y^{\mu-\nu-1}} \sum_{0\leq n\leq -\nu-1} \frac{1}{n!}\left(-\frac{b^2y}{2a}\right)^n K_{\mu-\nu-n-1}(ay).
\end{aligned}$$