THE INTEGRAL OF GEOMETRIC BROWNIAN MOTION

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Abstract

This paper is about the probability law of the integral of geometric Brownian motion over a finite time interval. A partial differential equation is derived for the Laplace transform of the law of the reciprocal integral, and is shown to yield an expression for the density of the distribution. This expression has some advantages over the ones obtained previously, at least when the normalized drift of the Brownian motion is a non-negative integer. Bougerol's identity and a relationship between Brownian motions with opposite drifts may also be seen to be special cases of these results.

Keywords: Brownian motion; Bougerol's identity; Asian options

1. Introduction

Let B stand for one-dimensional standard Brownian motion starting at 0, and define the "integral-exponential functional" of Brownian motion

$$A_t^{(\mu)} \;=\; \int_0^t e^{2\mu\tau+2B_\tau}\,d\tau, \qquad t\geq 0,\;\; \mu\in\mathbb{R}. \tag{1.1}$$

The main goal of this paper is to derive a new expression for the probability density function (p.d.f. in the sequel) of $A_t^{(\mu)}$. Some expressions have previously been obtained for this p.d.f.. Proposition 2, p. 527, in Yor (1992a) states that

$$\begin{aligned}\mathsf{P}(A_t^{(\mu)} \in du \,|\, B_t+\mu t = x) &= a_t(x,u)\,du\\ &= \frac{\sqrt{2\pi t}}{u}\exp\left(\frac{x^2}{2t}-\frac{1}{2u}(1+e^{2x})\right)\theta_{e^x/u}(t)\,du,\end{aligned} \tag{1.2}$$

where

$$\theta_r(t) \;=\; \frac{r}{\sqrt{2\pi^3 t}}\exp\left(\frac{\pi^2}{2t}\right)\int_0^\infty dy\,\exp(-y^2/2t)\exp(-r\cosh y)(\sinh y)\sin\left(\frac{\pi y}{t}\right).$$

The p.d.f. of $A_t^{(\mu)}$ results from integrating $a_t(x,u)$ as given above with respect to the normal probability density function with mean μt and variance t. Comtet *et al.* (1998) find the p.d.f. in the special case $\mu=0$ (formula (4.2) herein), and mention that it was also derived in Alili (1995). The following expression for the density of $A_t^{(\mu)}$, valid when $\mu<0$, is obtained in Monthus and Comtet (1994) and Comtet and Monthus (1996):

$$\begin{aligned}e^{-\frac{1}{2x}}\Bigg[2&\sum_{0\leq n<-\mu/2} e^{2tn(\mu+n)}\frac{(-1)^{n+1}(\mu+2n)}{\Gamma(1-\mu-n)}\left(\frac{1}{2x}\right)^{1-\mu-n}L_n^{-\mu-2n}\left(\frac{1}{2x}\right)\\ &+\frac{1}{2\pi^2}\int_0^\infty ds\,e^{-\frac{t}{2}(\mu^2+s^2)}s\sinh(\pi s)\left|\Gamma\left(\frac{\mu+is}{2}\right)\right|^2\left(\frac{1}{2x}\right)^{(1-\mu)/2}W_{(1-\mu)/2,is/2}\left(\frac{1}{2x}\right)\Bigg]\end{aligned} \tag{1.3}$$

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where $L_n^a(x)$ is the generalized Laguerre polynomial

$$L_n^a(x) \;=\; \frac{x^{-a}}{n!}e^x \frac{d}{dx^n}(x^{n+a}e^{-x}), \qquad n = 0,1,2\ldots$$

and $W_{p,q}$ is Whittaker's function. (Those authors write $+\mu$ where we write $-\mu$.)

Dufresne (2000) obtained a series representation for the p.d.f of $2A_t^{(\mu)}$:

$$g_\mu(t,y) \;=\; c^{a+1}y^{-b-2}e^{-c/y}\sum_{n=0}^{\infty} a_n(t)L_n^a(c/y), \tag{1.4}$$

where $a > -1$, $b \in \mathbb{R}$, $0 < c < 1$, and

$$\begin{aligned} a_n(t) \;&=\; \frac{n!}{\Gamma(n+a+1)}\mathsf{E}(2A_t^{(\mu)})^{b-a}L_n^a\left(\frac{c}{2A_t^{(\mu)}}\right) \\ &=\; \sum_{k=0}^{n}\frac{n!(-c)^k}{\Gamma(k+a+1)k!(n-k)!}\mathsf{E}(2A_t^{(\mu)})^{-(a-b+k)}. \end{aligned}$$

(*N.B.* Theorem 5.1 in Dufresne (2000) requires that $0 < c < e^{-2t\mu_-}$, but Corollary 2.3 below shows that (1.4) holds for $0 < c < 1$. Other results in Dufresne (2000) are similarly exended.)

Expression (1.2) is a double integral for the density. The author has not tried to evaluate (1.3) numerically, but the expression involves integrating the Whittaker function with respect to one its parameters, which means that the Whittaker function must be expressed either as an infinite sum or as an integral; the expression is in any case limited to negative μ. Expression (1.4) is an infinite combination of moments of $1/A_t^{(\mu)}$. For some particular values of μ these moments simplify somewhat, but otherwise each moment requires a separate numerical integration (see Dufresne, 2000, for details). The Laguerre series is thus an infinite sum of single integrals. This paper derives an alternative expression for $g_\mu(t,y)$ which is simpler than the ones above when $\mu \in \mathbb{N}$. The new expression should therefore be easier to use, in particular to compute truncated moments (as in financial applications). Whether it is an improvement (from a computational point of view) when $\mu \notin \mathbb{N}$ is unclear.

There have been a number of other papers on the same integral functional of Brownian motion, including Yor (1992a, 1992b, 1992c), De Schepper *et al.* (1992), Carmona *et al.* (1997), Dufresne (1999), Donati-Martin *et al.* (1999). Other references are given in those papers. The problem is directly related to the pricing of Asian options in the classical Black-Scholes framework, but the consequences of the results given here on option pricing are left to a subsequent article.

In Section 2 a partial differential equation (p.d.e. in the sequel) is derived for the Laplace transform of the density of $1/(2A_t^{(\mu)})$. Its solution is equivalent to the double transform obtained by Yor (1992c). One consequence of using the p.d.e. is that the double transform, and thus everything else in this paper, becomes independent of the theory of

Bessel processes. The p.d.e. is labeled "hypergeometric," as it involves the hypergeometric differential operator in one of the variables.

Section 3 examines a particular relationship between the solutions of the hypergeometric p.d.e. for different values of the drift (the "μ" in (1.1)). The relationship is a generalization of two other ones, the first an application of Girsanov's Theorem and time reversal to (1.1), the second a consequence of a property of beta and gamma distributions. Finally, the relationship is used in Section 4 to give an alternative expression for the p.d.f. of $A_t^{(\mu)}$; the latter involves the Hermite function H_μ and, as a result, simplifications occur when $\mu \in \mathbb{N}$.

Notation. The symbol "$\stackrel{\mathcal{L}}{=}$" means "have the same probability law." The hypergeometric and confluent hypergeometric functions (Lebedev, 1972, Chapter 9) are denoted ${}_2F_1(a,b,c;z)$ and ${}_1F_1(a,b;z)$, respectively.

2. The hypergeometric p.d.e. and the double transform

Define

$$h^{\mu,r}(s,t) \;=\; h(s,t) \;=\; e^{\mu^2 t/2}\mathsf{E}\,(2A_t^{(\mu)})^{-r}e^{s/(2A_t^{(\mu)})} \;=\; \mathsf{E}\,e^{\mu B_t}(2A_t^{(0)})^{-r}e^{s/(2A_t^{(0)})} \tag{2.1}$$

for $\mu \in \mathbb{R}$ (the last equality is a consequence of Girsanov's Theorem). Let $a_+ = \max(a,0)$, $a_- = \max(-a,0)$; the inequalities

$$t\exp(-2t\mu_- + 2\min_{\tau\in[0,t]} B_\tau) \;\le\; A_t^{(\mu)} \;\le\; t\exp(2t\mu_+ + 2\max_{\tau\in[0,t]} B_\tau) \tag{2.2}$$

imply that $\mathsf{E}(2A_t^{(\mu)})^{-r}$ is finite for $r\in\mathbb{C}$, and thus that $h^{\mu,r}(s,t)$ is finite for $\mu\in\mathbb{R}$, $r\in\mathbb{C}$, $\mathrm{Re}(s)\le 0$, $t>0$. As a function of (r,s), it is analytic in $\{r\in\mathbb{C}, \mathrm{Re}(s)<0\}$ (the domain where $h^{\mu,r}(s,t)$ is finite and analytic is seen to be larger in Corollary 2.3 below).

Remark. We will mostly deal with $\mu, r \in \mathbb{R}$, although several of the formulas are unchanged if μ or r is complex. Only in the proof of Theorem 4.2 is it required that $\mu\in\mathbb{C}$, essentially in order to use the fact that the p.d.f. of $A_t^{(\mu)}$ is an entire function of μ. The exponent in (2.1) is written "$s/(2A_t^{(\mu)})$" instead of the more common "$-s/(2A_t^{(\mu)})$," in order to make clear the connection with hypergeometric functions and differential equations. □

We now give a new proof as well as another expression for the double transform (2.6) (which was originally derived by Yor (1992c)). The double transform is the Laplace transform in t of the Mellin transform $\nu \mapsto \mathsf{E}\,(2A_t^{(\mu)})^\nu$. The result may be guessed from the following considerations. Using time reversal and Ito's formula (or otherwise),

$$\frac{d}{dt}\mathsf{E}\,(2A_t^{(\mu)})^\nu \;=\; a_\nu \mathsf{E}\,(2A_t^{(\mu)})^\nu + 2\nu\mathsf{E}\,(2A_t^{(\mu)})^{\nu-1},$$

with $a_\nu = 2\nu(\nu+\mu)$. Setting

$$m(\lambda,\nu) \;=\; \int_0^\infty e^{-\lambda t}\mathsf{E}\,(2A_t^{(\mu)})^\nu\,dt,$$

we find

$$m(\lambda,\nu) \;=\; \frac{2\nu}{\lambda-a_\nu}m(\lambda,\nu-1), \qquad \nu>0.$$

This recurrence equation implies

$$m(\lambda,\nu) \;=\; \frac{\Gamma(1+\nu)\Gamma\Big(-\frac{\mu}{2}+\frac{1}{2}\sqrt{2\lambda+\mu^2}-\nu\Big)\Gamma\Big(1+\frac{\mu}{2}+\frac{1}{2}\sqrt{2\lambda+\mu^2}\Big)}{\lambda\Gamma\Big(\frac{\mu}{2}+\frac{1}{2}\sqrt{2\lambda+\mu^2}+1+\nu\Big)\Gamma\Big(-\frac{\mu}{2}+\frac{1}{2}\sqrt{2\lambda+\mu^2}\Big)},$$

for $\nu=0,1,\ldots$ This is seen to agree with (2.5), from the usual formula

$$\frac{1}{\Gamma(c)}\,{}_2F_1(a,b,c;1) \;=\; \frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \mathrm{Re}(c-a-b)>0.$$

To use the above as a proper transform in λ and ν, the result must be extended to non-integral ν. The proof given below brings to light the important link between this problem and hypergeometric functions. The function $\overline{h}^{\mu,r}(s,\lambda)$ (see (2.4) below) may be seen as a triple transform in (r,s,λ), which extends $m(\lambda,-r)$.

Theorem 2.1. *For $\mu,r\in\mathbb{R}$, $h^{\mu,r}(s,t)$ satisfies the p.d.e.*

$$-\tfrac{1}{2}h_t \;=\; -\tfrac{1}{4}(\mu-2r)^2h+[r+(\mu-2r-1)s]h_s+s(1-s)h_{ss}, \tag{2.3}$$

(subscripts indicate partial derivatives) in $\{\mathrm{Re}(s)<0, t>0\}$, with $h(s,0+)=0$ for $\mathrm{Re}(s)<0$. The Laplace transform with respect to the variable t

$$\overline{h}^{\mu,r}(s,\lambda) \;=\; \overline{h}(s,\lambda) \;=\; \int_0^\infty e^{-\lambda t}h^{\mu,r}(s,t)\,dt \tag{2.4}$$

converges if $\sqrt{2\lambda}>\max(-\mu,\mu-2r)$ and if either (i) $\mathrm{Re}(s)<0$, $r\in\mathbb{R}$, or (ii) $\mathrm{Re}(s)=0$, $r<1$. If these conditions are fulfilled, then

$$\overline{h}^{\mu,r}(s,\lambda) \;=\; \frac{\Gamma(\alpha)\Gamma(\beta+r)}{2(1-s)^{\beta+r}\Gamma(\alpha+\beta+1)}\;{}_2F_1\Big(\alpha,\beta+r,\alpha+\beta+1;\frac{1}{1-s}\Big), \tag{2.5}$$

where $\alpha=\dfrac{\mu}{2}+\sqrt{\dfrac{\lambda}{2}}$, $\beta=\alpha-\mu$. Eq. (2.5) is equivalent to the following (Yor, 1992, théorème 2): let T_λ be an exponential random variable with mean $1/\lambda$, independent of B; then

$$2A^{(\mu)}_{T_\lambda} \;\stackrel{\mathcal{L}}{=}\; \frac{U}{G}, \tag{2.6}$$

where the variables $U\sim\mathrm{Beta}(1,\alpha_\mu)$, $G\sim\mathrm{Gamma}(\beta_\mu,1)$ are independent, $\alpha_\mu=\dfrac{\mu}{2}+\dfrac{1}{2}\sqrt{2\lambda+\mu^2}$, and $\beta_\mu=\alpha_\mu-\mu$.

Proof. Let $\mu, r \in \mathbb{R}$. If $s < 0$, then (2.2) implies

$$\mathsf{E}\, e^{\mu B_t}(2A_t^{(0)})^{-r} e^{s/(2A_t^{(0)})} \ \le\ \mathsf{E}\, e^{\mu B_t}(2A_t^{(0)})^{-r} \exp\left(\frac{s}{2t} e^{-2\max_{\tau\in[0,t]} B_\tau}\right),$$

and this leads to $h(s, 0+) = 0$. Now let $\mathrm{Re}(s) < 0$. For fixed $t > 0$, time reversal yields the identity in law (Dufresne, 1989)

$$Y_t \ = \ e^{2\mu t + 2B_t} \int_0^t e^{-(2\mu\tau + 2B_\tau)}\, d\tau \ \stackrel{\mathcal{L}}{=}\ A_t^{(\mu)}. \tag{2.7}$$

From Ito's formula,

$$\begin{aligned} dY_t &= [(2\mu + 2)Y_t + 1]\, dt + 2Y_t\, dB_t \\ Y_t^{-r} e^{s/(2Y_t)} &= Y_{t_0}^{-r} e^{s/(2Y_{t_0})} + \int_{t_0}^t a(Y_\tau)\, d\tau + \int_{t_0}^t b(Y_\tau)\, dB_\tau, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} a(y) &= [(2\mu + 2)y + 1] \frac{d}{dy}\left(y^{-r} e^{s/(2y)}\right) + 2y^2 \frac{d^2}{dy^2}\left(y^{-r} e^{s/(2y)}\right) \\ &= \left[\frac{1}{2}(s^2 - s) y^{-2-r} + [-r + (1 + 2r - \mu)s] y^{-1-r} + 2r(r - \mu) y^{-r}\right] e^{s/(2y)} \\ b(y) &= 2y \frac{d}{dy}\left(y^{-r} e^{s/(2y)}\right) \ = \ -(2ry^{-r} + sy^{-r-1}) e^{s/(2y)}. \end{aligned}$$

Taking expectations on both sides of (2.8), and noting that

$$\mathsf{E} \int_0^t b^2(Y_\tau)\, d\tau \ < \ \infty \qquad \text{implies} \qquad \mathsf{E} \int_0^t b(Y_\tau)\, dB_\tau \ = \ 0,$$

it is seen that

$$\begin{aligned} &\frac{d}{dt} \mathsf{E}\, Y_t^{-r} e^{s/(2Y_t)} \\ &\quad = \mathsf{E}\left[\frac{1}{2}(s^2 - s) Y_t^{-2-r} + [-r + (1 + 2r - \mu)s] Y_t^{-1-r} + 2r(r - \mu) Y_t^{-r}\right] e^{s/(2Y_t)} \\ &\quad = \left[2(s^2 - s)\frac{d^2}{ds^2} + 2[-r + (1 + 2r - \mu)s]\frac{d}{ds} + 2r(r - \mu)\right] \mathsf{E}\, Y_t^{-r} e^{s/(2Y_t)}. \end{aligned} \tag{2.9}$$

We get (2.3) from

$$\frac{d}{dt}\left[e^{\mu^2 t/2} \mathsf{E}\, Y_t^{-r} e^{s/(2Y_t)}\right] \ = \ \frac{\mu^2}{2} e^{\mu^2 t/2} \mathsf{E}\, Y_t^{-r} e^{s/(2Y_t)} + e^{\mu^2 t/2} \frac{d}{dt} \mathsf{E}\, Y_t^{-r} e^{s/(2Y_t)}.$$

Next, we show that $\bar{h}(s,\lambda)$ is finite if λ is large enough. If $r<0$, then (2.2) implies

$$\mathsf{E}\,(A_t^{(\mu)})^{-r} \;\le\; t^{-r}e^{-(2rt\mu_+)}\mathsf{E}\,e^{-2r\sqrt{t}|B_1|} \le\; t^{-r}e^{-(2rt\mu_+)}(e^{2r^2t}+\tfrac{1}{2}).$$

Similarly, for $r\ge 0$,

$$\mathsf{E}\,(A_t^{(\mu)})^{-r} \;\le\; t^{-r}e^{(2rt\mu_-)}(e^{2r^2t}+\tfrac{1}{2}).$$

We conclude that the integral

$$\int_0^\infty e^{-\lambda t}\mathsf{E}\,(2A_t^{(\mu)})^{-r}\,dt$$

converges if $r<1$ and λ is greater than some λ_0, which means that $\bar{h}^{\mu,r}(0,\lambda)$ is finite if $\lambda>\lambda_1=\lambda_0+\mu^2/2$ and $r<1$. Similar arguments show that $\bar{h}^{\mu,r}(s,\lambda)$ is finite if $r\in\mathbb{R}$, $\mathrm{Re}(s)<0$ and λ is larger than some λ_2.

For $\mathrm{Re}(s)<0$ and $\lambda>\lambda_2$, multiply (2.3) by $e^{-\lambda t}$, and then integrate to obtain

$$0 \;=\; [\tfrac{\lambda}{2}-\tfrac{1}{4}(\mu-2r)^2]\bar{h}+[r-(1+2r-\mu)s]\bar{h}_s+s(1-s)\bar{h}_{ss}. \tag{2.10}$$

This is an instance of the Gauss hypergeometric differential equation

$$0 \;=\; -abG+[c-(1+a+b)s]G_s+s(1-s)G_{ss}, \tag{2.11}$$

which has general solution (if $c\notin\mathbb{N}$)

$$G(s) \;=\; C_1\,{}_2F_1(a,b,c;s)+C_2s^{1-c}\,{}_2F_1(1-c+a,1-c+b,2-c;s), \tag{2.12}$$

where C_1, C_2 are constants. The change of variables

$$z=1/(1-s), \qquad g(z)=z^{-a}G(s), \tag{2.13}$$

transforms Eq. (2.11) into

$$0 \;=\; a(b-c)g+[1+a-b-(1+a+c-b)z]g_z+z(1-z)g_{zz},$$

which is of the same type as (2.11), but with new parameters $a'=a$, $b'=c-b$, $c'=1+a-b$. Now consider the case $r=0$; in (2.10), the parameters become

$$a \;=\; -\frac{\mu}{2}+\sqrt{\frac{\lambda}{2}}, \qquad b \;=\; -\frac{\mu}{2}-\sqrt{\frac{\lambda}{2}}, \qquad c \;=\; 0.$$

Applying the transformation (2.13) to $G(s)=\bar{h}^{\mu,0}(s,\lambda)$, we get

$$a' = \; -\frac{\mu}{2}+\sqrt{\frac{\lambda}{2}}, \qquad b' \;=\; \frac{\mu}{2}+\sqrt{\frac{\lambda}{2}}, \qquad c' \;=\; 1+\sqrt{2\lambda}.$$

From (2.12), the general solution of (2.10) (when $r = 0$) may be written

$$\begin{aligned}\overline{h}^{\mu,0}(s,\lambda) &= C_1(1-s)^{-a'}\,{}_2F_1\Big(a',b',c';\frac{1}{1-s}\Big)\\ &\quad + C_2(1-s)^{c'-a'-1}\,{}_2F_1\Big(1-c'+a',1-c'+b',2-c';\frac{1}{1-s}\Big),\end{aligned}$$

C_2 must be 0, since $\overline{h}^{\mu,0}(s,\lambda) \to 0$ as $s \to -\infty$. Letting $s = 0$, we then find (Lebedev, 1972, p. 244)

$$\begin{aligned}C_1 &= \frac{\overline{h}^{\mu,0}(0,\lambda)}{{}_2F_1(a',b',c',1)} = \overline{h}^{\mu,0}(0,\lambda)\frac{\Gamma(c'-a')\Gamma(c'-b')}{\Gamma(c')\Gamma(c'-a'-b')}\\ &= \overline{h}^{\mu,0}(0,\lambda)\frac{\Gamma\Big(1+\frac{\mu}{2}+\sqrt{\frac{\lambda}{2}}\Big)\Gamma\Big(1-\frac{\mu}{2}+\sqrt{\frac{\lambda}{2}}\Big)}{\Gamma\big(1+\sqrt{2\lambda}\big)}.\end{aligned}$$

But

$$\overline{h}^{\mu,0}(0,\lambda) = \int_0^\infty e^{-\lambda t}\,\mathsf{E}\,e^{\mu B_t}\,dt = \frac{2}{2\lambda-\mu^2},$$

and so

$$\begin{aligned}\overline{h}^{\mu,0}(s,\lambda) &= \frac{2}{2\lambda-\mu^2}\frac{\Gamma\Big(1+\frac{\mu}{2}+\sqrt{\frac{\lambda}{2}}\Big)\Gamma\Big(1-\frac{\mu}{2}+\sqrt{\frac{\lambda}{2}}\Big)}{(1-s)^{a'}\Gamma\big(1+\sqrt{2\lambda}\big)}\,{}_2F_1\Big(a',b',c';\frac{1}{1-s}\Big)\\ &= \frac{\Gamma\Big(\frac{\mu}{2}+\sqrt{\frac{\lambda}{2}}\Big)\Gamma\Big(-\frac{\mu}{2}+\sqrt{\frac{\lambda}{2}}\Big)}{2(1-s)^{a'}\Gamma\big(1+\sqrt{2\lambda}\big)}\,{}_2F_1\Big(a',b',c';\frac{1}{1-s}\Big). \qquad (2.14)\end{aligned}$$

The last expression is (2.5) when $r = 0$. The formula for arbitrary r will follow after we prove (2.6). The latter will be achieved (for all r) if it is shown that for some particular r and for all $\mathrm{Re}(s) \le 0$,

$$\int_0^\infty \lambda e^{-\lambda t}\,\mathsf{E}\,(2A_t^{(\mu)})^{-r}e^{s/(2A_t^{(\mu)})}\,dt = \mathsf{E}\left(\frac{G}{U}\right)^r e^{sG/U},$$

or, equivalently, that

$$\lambda\overline{h}^{\mu,r}(s,\lambda+\mu^2/2) = \mathsf{E}\left(\frac{G}{U}\right)^r e^{sG/U}.$$

Now, for $\mathrm{Re}(s) \le 0$,

$$\mathsf{E}\left(\frac{G}{U}\right)^r e^{sG/U}$$

$$
\begin{aligned}
&= \frac{\alpha_\mu}{\Gamma(\beta_\mu)} \int_0^1 du\, u^{-r}(1-u)^{\alpha_\mu-1} \int_0^\infty dx\, x^{\beta_\mu+r-1} e^{-x+sx/u} \\
&= \frac{\alpha_\mu \Gamma(\beta_\mu+r)}{\Gamma(\beta_\mu)} \int_0^1 du\, u^{-r}(1-u)^{\alpha_\mu-1} \left(1-\frac{s}{u}\right)^{-\beta_\mu-r} \\
&= \frac{\alpha_\mu \Gamma(\beta_\mu+r)}{\Gamma(\beta_\mu)} \int_0^1 du\, \frac{u^{\beta_\mu}(1-u)^{\alpha_\mu-1}}{(u-s)^{\beta_\mu+r}} \\
&= \frac{\alpha_\mu \Gamma(\beta_\mu+r)}{\Gamma(\beta_\mu)} \int_0^1 dv\, \frac{(1-v)^{\beta_\mu} v^{\alpha_\mu-1}}{[(1-s)-v]^{\beta_\mu+r}} \\
&= \frac{\alpha_\mu \Gamma(\beta_\mu+r)}{\Gamma(\beta_\mu)(1-s)^{\beta_\mu+r}} \int_0^1 dv\, v^{\alpha_\mu-1}(1-v)^{\beta_\mu} \left(1-\frac{v}{1-s}\right)^{-\beta_\mu-r} \\
&= \frac{\alpha_\mu \Gamma(\beta_\mu+r)}{\Gamma(\beta_\mu)(1-s)^{\beta_\mu+r}} \frac{\Gamma(\alpha_\mu)\Gamma(\beta_\mu+1)}{\Gamma(\alpha_\mu+\beta_\mu+1)}\, {}_2F_1\Big(\alpha_\mu, \beta_\mu+r, \alpha_\mu+\beta_\mu+1; \frac{1}{1-s}\Big) \\
&= \frac{\lambda \Gamma(\alpha_\mu)\Gamma(\beta_\mu+r)}{2(1-s)^{\beta_\mu+r}\Gamma(\alpha_\mu+\beta_\mu+1)}\, {}_2F_1\Big(\alpha_\mu, \beta_\mu+r, \alpha_\mu+\beta_\mu+1; \frac{1}{1-s}\Big),
\end{aligned}
$$

where we have used the integral representation of the hypergeometric function (Lebedev, 1972, p. 240). Comparing with (2.14) above (in the special case $r = 0$, and with λ replaced with $\lambda + \mu^2/2$) finishes the proof of (2.6). It is now possible to be more precise regarding the set of λ such that (2.4) converges. If $s < 0$, then $h^{\mu,r}(s,t)$ is positive and finite, and the function $\lambda \mapsto \overline{h}^{\mu,r}(s,\lambda)$ has to have a singularity at the infimum of the set of points where (2.4) converges (Widder, 1946, p. 58). For $|\arg(1-z)| < \pi$, the function

$$
(u,v,w) \mapsto \frac{\Gamma(u)\Gamma(v)}{\Gamma(w)}\, {}_2F_1(u,v,w;z)
$$

is analytic except for poles at $u, v = 0, -1, -2, \ldots$; hence (2.5) holds if α and $\beta + r$ are strictly positive, that is, if $\sqrt{2\lambda} > \max(-\mu, \mu - 2r)$, and if $\mathrm{Re}(s) < 0$. Now turn to the case $\mathrm{Re}(s) = 0$. In order for $\overline{h}^{\mu,r}(0,\lambda)$ to be finite, $\overline{h}^{\mu,r}(s,\lambda)$ must be finite for all $s < 0$, which requires $\sqrt{2\lambda} > \max(-\mu, \mu - 2r)$. Then

$$
\begin{aligned}
\lim_{s\to 0-} \overline{h}^{\mu,r}(s,\lambda) &= \lim_{s\to 0-} \frac{\Gamma(\alpha)\Gamma(\beta+r)}{2(1-s)^{\beta+r}\Gamma(\alpha+\beta+1)}\, {}_2F_1\Big(\alpha, \beta+r, \alpha+\beta+1, \frac{1}{1-s}\Big) \\
&= \frac{\Gamma(\alpha)\Gamma(\beta+r)}{2\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+1)\Gamma(1-r)}{\Gamma(\alpha+1-r)\Gamma(\beta+1)} \\
&= \frac{\Gamma(\alpha)\Gamma(\beta+r)\Gamma(1-r)}{2\Gamma(\alpha+1-r)\Gamma(\beta+1)} \;<\; \infty
\end{aligned}
$$

if $r < 1$. If $r > 1$, then (Lebedev, 1972, p.248, Eq. (9.5.3))

$$\begin{aligned}
&\lim_{s\to 0-} \overline{h}^{\mu,r}(s,\lambda) \\
&\quad = \lim_{s\to 0-} \frac{\Gamma(\alpha)\Gamma(\beta+r)}{2(1-s)^{\beta+r}\Gamma(\alpha+\beta+1)} \left(\frac{s}{s-1}\right)^{1-r} {}_2F_1\Big(\alpha+1-r,\beta+1,\alpha+\beta+1,\frac{1}{1-s}\Big) \\
&\quad = \lim_{s\to 0-} \frac{\Gamma(\alpha)\Gamma(\beta+r)}{2\Gamma(\alpha+\beta+1)} \left(\frac{s}{s-1}\right)^{1-r} \frac{\Gamma(\alpha+\beta+1)\Gamma(r-1)}{\Gamma(\alpha)\Gamma(\beta+r)} \;=\; \infty,
\end{aligned}$$

and so $\overline{h}^{\mu,r}(0,\lambda)=\infty$. □

Remark. It can be shown that $A_t^{(\mu)}$ converges almost surely when $t\to\infty$ if, and only if, $\mu<0$. If $\mu<0$, then replacing the left-hand side of (2.9) with 0 and solving constitutes one more way of proving that $1/(2A_\infty^{(\mu)})$ has a Gamma$(-\mu,1)$ distribution. □

The next result is an application of a quadratic transformation formula for hypergeometric functions.

Corollary 2.2. *The following formulas hold (*$\mathrm{Re}(s)\le 0$*):*

$$\overline{h}^{0,\frac12}(s,\lambda) \;=\; \frac{\sqrt{\pi}}{\sqrt{2\lambda(1-s)}}\left(\sqrt{-s}+\sqrt{1-s}\right)^{-\sqrt{2\lambda}} \tag{2.15}$$

$$h^{0,\frac12}(s,t) \;=\; \frac{\exp\{-(\operatorname{arcsinh}^2\sqrt{-s})/2t\}}{\sqrt{2t(1-s)}} \tag{2.16}$$

$$\overline{h}^{1,\frac12}(s,\lambda) \;=\; \frac{\sqrt{\pi}}{\sqrt{2\lambda}}\left(\sqrt{-s}+\sqrt{1-s}\right)^{-\sqrt{2\lambda}} \tag{2.17}$$

$$h^{1,\frac12}(s,t) \;=\; \frac{\exp\{-(\operatorname{arcsinh}^2\sqrt{-s})/2t\}}{\sqrt{2t}}. \tag{2.18}$$

Proof. To obtain (2.15), apply formula (9.6.2), p. 251 in Lebedev (1972),

$$\begin{aligned}
&{}_2F_1(a,b,a+b+\tfrac12;z) \;= \\
&\qquad \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2a} {}_2F_1\left(2a,a-b+\tfrac12,a+b+\tfrac12;\frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}\right),
\end{aligned} \tag{2.19}$$

which is valid if $|\arg(1-z)|<\pi$ and $-(a+b+\frac12)\notin\mathbb{N}$. First, let $\mu=0$, $r=\frac12$ in (2.5):

$$\begin{aligned}
\overline{h}^{0,\frac12}(s,\lambda) &= \frac{\Gamma\Big(\sqrt{\frac{\lambda}{2}}\Big)\Gamma\Big(\frac12+\sqrt{\frac{\lambda}{2}}\Big)}{2(1-s)^{\sqrt{\frac{\lambda}{2}}+\frac12}\Gamma\big(1+\sqrt{2\lambda}\big)}\,{}_2F_1\Big(\sqrt{\tfrac{\lambda}{2}},\tfrac12+\sqrt{\tfrac{\lambda}{2}},1+\sqrt{2\lambda};\frac{1}{1-s}\Big) \\
&= \frac{\sqrt{\pi}}{2^{\sqrt{2\lambda}}(1-s)^{\sqrt{\frac{\lambda}{2}}+\frac12}\sqrt{2\lambda}}2^{\sqrt{2\lambda}}\left(1+\sqrt{\frac{-s}{1-s}}\right)^{-\sqrt{2\lambda}} \\
&= \frac{\sqrt{\pi}}{(1-s)^{\frac12}\sqrt{2\lambda}}\left(\sqrt{-s}+\sqrt{1-s}\right)^{-\sqrt{2\lambda}},
\end{aligned}$$

where we have used the duplication property of the gamma function and the identity ${}_2F_1(a, 0, c; z) = 1$. Now

$$\operatorname{arcsinh} z \;=\; \log(z + \sqrt{1+z^2}) \tag{2.20}$$

and so

$$\bar{h}^{0,\frac{1}{2}}(s, \lambda) \;=\; \frac{\sqrt{\pi}}{(1-s)^{\frac{1}{2}}\sqrt{2\lambda}} \exp(-\sqrt{2\lambda}\operatorname{arcsinh}\sqrt{-s}).$$

This expression is then inverted (from λ to t) to give (2.16) (use the formula

$$\int_0^\infty \frac{e^{-\lambda t - z^2/2t}}{\sqrt{2t}}\,dt \;=\; \sqrt{\frac{\pi}{2\lambda}}e^{-z\sqrt{2\lambda}},$$

which is valid for $0 \le \arg z \le \frac{\pi}{4}$.) Eqs. (2.17) and (2.18) result from the same steps, letting $\mu = 1$, $r = 1/2$ in (2.5) and then applying (2.19). □

Formula (2.16) was shown by Yor (1992a, Eq. (1e)) to be equivalent to Bougerol's identity (Bougerol, 1983): if W is a standard Brownian motion independent of B, then for fixed $t > 0$,

$$\int_0^t e^{B_\tau}\,dW_\tau \;\stackrel{\mathcal{L}}{=}\; \sinh B_t.$$

Bougerol's identity can thus be seen as a degenerate case, where a certain hypergeometric series terminates after just one term; Eq. (2.18) is the "companion" case where the first two parameters of the hypergeometric function (2.5) are reversed. There are other ways of obtaining Bougerol's identity, in addition to the original derivation, see Yor (1992a, Section 3) and Alili *et al.* (1997). Corollary 2.2 is also related to the following identity (Alili *et al.* 1997, p. 10, Eq. (15)), valid for $\mu, x \in \mathbb{R}$:

$$\begin{aligned} h^{\mu,\frac{1}{2}}(-x^2, t) \;&=\; \frac{1}{\sqrt{2t}}(1+x^2)^{(\mu-1)/2}\exp\left(-\frac{\operatorname{arcsinh}^2 x}{2t}\right) \\ &\quad \times \mathsf{E}\left[\exp\left(-\frac{\mu-\mu^2}{2}\int_0^t \frac{ds}{\cosh^2 B_s}\right)\,\middle|\, \sinh B_t = x\right]. \end{aligned}$$

Bougerol's identity is obtained upon setting $\mu = 0$ (Eq. (2.16)); the other case where a comparable simplification occurs is $\mu = 1$ (Eq. (2.18)).

Corollary 2.3. *The function $h^{\mu,r}(s,t)$ is finite, and the p.d.e. (2.3) holds, for $r \in \mathbb{R}$,* $\mathrm{Re}(s) < 1$*,* $t > 0$*. More precisely,*

$$\sup\{s \in \mathbb{R} \,|\, h^{\mu,r}(s,t) < \infty\} \;=\; 1, \qquad \mu \in \mathbb{R}, \quad t > 0.$$

Proof. The exponent on the right hand side of (2.16) may be rewritten

$$-(\operatorname{arcsinh}^2\sqrt{-s})/(2t) \;=\; s\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, s)^2/(2t) \tag{2.21}$$

where the Gauss hypergeometric function ${}_2F_1(\frac{1}{2},\frac{1}{2},\frac{3}{2},s)$ has a convergent power series about 0 in s, with convergence radius equal to 1. Thus $h^{0,1/2}(s,t)$ is analytic in $\{\mathrm{Re}(s)<1\}$. Let $X = 1/(2A_t^{(0)})$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, and apply Hölder's inequality: if $q\mathrm{Re}(s)<1$, then

$$|h^{\mu,r}(s,t)| \;\le\; \Big[\mathsf{E}|e^{\mu p B_t}X^{pr-\frac{p}{2q}}|\Big]^{\frac{1}{p}}\Big[\mathsf{E}|X^{\frac{1}{2}}e^{qsX}|\Big]^{\frac{1}{q}} \;<\infty.$$

Similarly, if there were μ,r,s such that $h^{\mu,r}(s,t)<\infty$, $s>1$, then

$$\mathsf{E}\,X^{\frac{1}{2}}e^{\frac{s}{q}X} \;\le\; \Big[\mathsf{E}\,e^{-\frac{\mu p}{q}B_t}X^{p(\frac{1}{2}-\frac{r}{q})}\Big]^{\frac{1}{p}}\Big[\mathsf{E}\,e^{\mu B_t}X^r e^{sX}\Big]^{\frac{1}{q}}$$

would lead to a contradiction (just let $1<q\le s$). □

3. Relationships among $\{\mathbf{h}^{\mu,\mathbf{r}}\}$ for different drifts

Theorem 3.1. *For all $\mu,r\in\mathbb{R}$, $\mathrm{Re}(s)<1$, $t>0$,*

$$h^{\mu,r}(s,t) \;=\; (1-s)^{\mu-r}h^{2r-\mu,r}(s,t). \tag{3.1}$$

Proof. One proof consists in checking directly that the right-hand side of (3.1) satisfies the p.d.e. (2.3). Both functions have Laplace transforms in t, and are equal when $t=0$, $s=0$ or when $s\to-\infty$. The proof of Theorem 2.1 thus shows that the two sides of (3.1) have the same Laplace transform in t, for any $r<1$. The result for arbitrary r follows by analytic continuation. Another justification is to let $\mu'=2r-\mu$ in (2.5), and then note that ${}_2F_1(a,b,c;z)={}_2F_1(b,a,c;z)$. □

Theorem 3.1 is an extension of the Corollaries 3.2 and 3.3, which had already been obtained by very different means. The first Corollary is an immediate consequence of the time reversal identity (2.7) and of Girsanov's Theorem, while the second one results from the double transform (2.6) and a property of beta and gamma distributions (Dufresne, 1997; see also Matsumoto & Yor, 1998, 1999).

Corollary 3.2. *For all $\mu,r\in\mathbb{R}$,* $\mathsf{E}\,(2A_t^{(\mu)})^{-r} \;=\; e^{2r(r-\mu)t}\mathsf{E}\,(2A_t^{(2r-\mu)})^{-r}.$

Corollary 3.3. *If $G_\mu\sim\mathrm{Gamma}(\mu,1)$ is independent of $A_t^{(\mu)}$, then for $\mu,t>0$,*

$$\frac{1}{2A_t^{(\mu)}}+G_\mu \;\stackrel{\mathcal{L}}{=}\; \frac{1}{2A_t^{(-\mu)}}.$$

Corollary 3.4. *Let $(a)_0=1$, $(a)_k=a(a+1)\cdots(a+k-1)$, $k\in\mathbb{N}_+$. For $\mu,r\in\mathbb{R}$, $n\in\mathbb{N}$, $t>0$,*

$$e^{\mu^2t/2}\mathsf{E}\,(2A_t^{(\mu)})^{-r-n} \;=\; \sum_{k=0}^{n}\binom{n}{k}(r-\mu)_k\,e^{(2n-2k+\mu)^2t/2}\mathsf{E}\,(2A_t^{(2n-2k+\mu)})^{-r-n+k}$$

$$\mathsf{E}\,(2A_t^{(\mu+2)})^{-1} \;=\; e^{-(2\mu+2)t}\Big[\mu+\mathsf{E}\,(2A_t^{(\mu)})^{-1}\Big]$$

$$\mathsf{E}\,(2A_t^{(\mu+2)})^{-\mu-1} \;=\; e^{-(2\mu+2)t}\mathsf{E}\,(2A_t^{(\mu)})^{-\mu-1}.$$

Proof. From Theorem 3.1 ($\mathrm{Re}(s) < 1$)

$$\begin{aligned} h^{\mu,r+n}(s,t) &= \frac{\partial^n}{\partial s^n} h^{\mu,r}(s,t) \;=\; \frac{\partial^n}{\partial s^n}\left[(1-s)^{\mu-r} h^{2r-\mu,r}(s,t)\right] \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k}{\partial s^k}(1-s)^{\mu-r} \frac{\partial^{n-k}}{\partial s^{n-k}} h^{2r-\mu,r}(s,t) \\ &= \sum_{k=0}^{n} \binom{n}{k} (r-\mu)_k (1-s)^{\mu-r-k} h^{2r-\mu,r+n-k}(s,t) \\ &= \sum_{k=0}^{n} \binom{n}{k} (r-\mu)_k (1-s)^{-n} h^{2n-2k+\mu,r+n-k}(s,t). \end{aligned}$$

We then let $s = 0$. The other formulas result from letting $n = 1$, with $r = 0$ first and then $r = \mu$. □

Theorem (3.1) also yields recursive relationships for the p.d.f. of $2A_t$, for example (4.22) below.

4. The probability density function of $\mathbf{A_t^{(\mu)}}$

Let $f_\mu(x,t)$ be the density of $1/\big(2A_t^{(\mu)}\big)$, and define

$$p_\mu(x,t) \;=\; e^{\mu^2 t/2} f_\mu(x,t).$$

The usual inversion integral for Laplace transforms is

$$p_\mu(x,t) \;=\; \frac{x^{-r}}{2\pi i} \int_\Delta e^{sx} h^{\mu,r}(-s,t)\, ds, \tag{4.1}$$

where the path of integration Δ goes from $c - i\infty$ to $c + i\infty$, with $c \in \mathbb{R}$ such that the path of integration is wholly to the right of the singularities of $h^{\mu,r}(-s,t)$.

Formula (4.2) below agrees with Eq. (21) of Comtet *et al.* (1998).

Theorem 4.1. *For $t, x > 0$,*

$$\begin{aligned} p_0(x,t) &= x^{-\frac{1}{2}} \int_\Delta e^{sx} K(s,t)\, ds \\ &= 2x^{-\frac{1}{2}} \int_0^\infty e^{-x\cosh^2 y} q(y,t) \cos\left(\frac{\pi y}{2t}\right) dy \end{aligned} \tag{4.2}$$

$$\begin{aligned} p_1(x,t) &= x^{-\frac{1}{2}} \int_\Delta e^{sx} K(s,t)\sqrt{1+s}\, ds \\ &= 2x^{-\frac{1}{2}} \int_0^\infty e^{-x\cosh^2 y} q(y,t) \sinh y \, \sin\left(\frac{\pi y}{2t}\right) dy, \end{aligned} \tag{4.3}$$

where

$$K(s,t) \;=\; \frac{e^{-(\operatorname{arcsinh}^2\sqrt{s})^2/2t}}{2\pi i\sqrt{2t(1+s)}}, \qquad q(y,t) \;=\; \frac{e^{\frac{\pi^2}{8t}-\frac{y^2}{2t}}}{\pi\sqrt{2t}}\cosh y\,.$$

Moreover, for $\mu = 0, 1$ and each $t > 0$ the function $x \mapsto p_\mu(x,t)$ is $C^\infty(\mathbb{R}_+)$.

Proof. Formulas (4.2) and (4.3) are obtained by modifying the path of integration in the inversion integral (4.1). We consider p_0 in detail, the other case is similar.

By (2.21), we see that $s \mapsto K(s,t)$ has a branch cut $(-\infty,-1]$, and no other singularity; we thus set the path Δ to the right of $s=-1$. From (2.20),

$$\lim_{|s|\to\infty} sK(s,t) \;=\; 0$$

uniformly with respect to $\arg s \in (-\pi,\pi)$, and thus the path of integration Δ may be replaced with:

$$\Delta_{\theta,\varepsilon} \;=\; \{-1+\rho e^{-i\theta},\ \varepsilon\le\rho<\infty;\ -1+\varepsilon e^{i\phi},\ -\theta<\phi<\theta;\ -1+\rho e^{i\theta},\ \varepsilon\le\rho<\infty\}$$

with $\frac{\pi}{2} < \theta < \pi$, $\varepsilon > 0$. We then let θ increase to π, and ϵ decrease to 0. In the limit the contribution of the integral along the circle around -1 is nil, since $\operatorname{arcsinh} z$ is uniformly bounded in a neighborhood of $z=-1$ and, as a consequence,

$$\lim_{z\to 0} zK(z-1) \;=\; 0.$$

(Observe that

$$\operatorname{arcsinh}\sqrt{\varepsilon e^{i\theta}-1} \;=\; \log(\sqrt{\varepsilon e^{i\theta}-1}+\sqrt{\varepsilon e^{i\theta}})$$

tends to $\pm\frac{i\pi}{2}$ as ε decreases to 0, depending on the sign of $\theta \in (-\pi,\pi)$.) We are thus left with the integral along the path $\Delta_{\pi,0}$, which goes from $-\infty$ to -1 just below the real line, and from -1 to $+\infty$ just above it. Hence

$$\begin{aligned} x^{\frac{1}{2}}\,p_0(x,t) \;&=\; \int_1^\infty e^{-ux}[K(ue^{-i\pi},t)-K(ue^{i\pi},t)]\,du \\ &=\; \int_0^\infty e^{-(v+1)x}[K((v+1)e^{-i\pi},t)-K((v+1)e^{i\pi},t)]\,dv. \end{aligned}$$

Now $\sqrt{1+(v+1)e^{\pm i\pi}} = \pm i\sqrt{v}$ and

$$\begin{aligned} \operatorname{arcsinh}\sqrt{(v+1)e^{\pm i\pi}} \;&=\; \log(\sqrt{(v+1)e^{\pm i\pi}}+\sqrt{1+(v+1)e^{\pm i\pi}}) \\ &=\; \log(\sqrt{v+1}+\sqrt{v}) \pm \frac{i\pi}{2} \\ &=\; \operatorname{arcsinh}\sqrt{v} \pm \frac{i\pi}{2}. \end{aligned}$$

Therefore

$$
\begin{aligned}
p_0(x,t) \\
&= \frac{e^{\pi^2/(8t)}}{\pi\sqrt{2tx}}\int_0^\infty \frac{1}{\sqrt{v}} e^{-(v+1)x-(\mathrm{arcsinh}\sqrt{v})^2/(2t)}\frac{1}{2}[e^{i\pi \mathrm{arcsinh}\sqrt{v}/(2t)}+e^{-i\pi\mathrm{arcsinh}\sqrt{v}/(2t)}]\,dv \\
&= \frac{e^{\pi^2/(8t)}}{\pi\sqrt{2tx}}\int_0^\infty \frac{1}{\sqrt{v}} e^{-(v+1)x-(\mathrm{arcsinh}\sqrt{v})^2/(2t)}\cos\left(\frac{\pi}{2t}\mathrm{arcsinh}\sqrt{v}\right)dv \\
&= \frac{2\,e^{\pi^2/(8t)}}{\pi\sqrt{2tx}}\int_0^\infty e^{-x\cosh^2 y - y^2/(2t)}\cosh y\,\cos(\tfrac{\pi y}{2t})\,dy.
\end{aligned}
$$

In the case of p_1, the reasoning is the same, the only difference being the factor $\sqrt{1+s} = \sqrt{1+(v+1)e^{\pm i\pi}} = \pm i\sqrt{v} = \pm i\sinh y$, which has the effect of replacing $\cos(\frac{\pi y}{2t})$ with $\sin(\frac{\pi y}{2t})\sinh y$ in the last integral. □

Remark. It is interesting to note that for $\mu = 0, 1$ the p.d.f. of $1/\big(2A_t^{(\mu)}\big)$ may be expressed as a single integral, which is straightforward to compute numerically for specific values of x; this is a direct consequence of the simplifications which occur in those two cases, as seen in Corollary 2.2. Observe that one factor in the integrand is a circular function with argument $\pi y/2t$, which makes numerical integration more difficult when t is small. The same holds for other values of μ, see Theorem 4.2 below. The consequence is that it is harder to obtain the density of $A_t^{(\mu)}$ numerically for short durations t than for longer ones, a fact which is *a priori* counterintuitive, but had already been noted in connection with the Laguerre expansion of the density (see Dufresne, 2000). □

Now turn to other values of μ. First, consider the cases where $\mu = 2n$, $n \in \mathbb{N}_+$. From Theorem 3.1,

$$h^{0,n}(s,t) \;=\; (1-s)^{-n}h^{2n,n}(s,t),$$

which implies

$$x^n e^x p_0(x,t) \;=\; \frac{1}{\Gamma(n)}\int_0^x (x-y)^{n-1}e^y y^n p_{2n}(y,t)\,dy,$$

whence

$$
\begin{aligned}
p_{2n}(x,t) &= x^{-n}e^{-x}\frac{d^n}{dx^n}\left[e^x x^n p_0(x,t)\right] \\
&= x^{-n}e^{-x}\int_\Delta K(s,t)\left(\frac{d^n}{dx^n}[x^{n-\frac{1}{2}}e^{(1+s)x}]\right)ds.
\end{aligned}
$$

For $a > -1$, $n = 0, 1, \ldots$,

$$\frac{d^n}{dx^n}[x^{n+a}e^{-bx}] \;=\; b^{-a}\frac{d^n}{dy^n}[y^{n+a}e^{-y}]\bigg|_{y=bx} \;=\; x^a e^{-bx}n!L_n^a(bx).$$

Thus

$$p_{2n}(x,t) \;=\; x^{-n-\frac{1}{2}}\int_\Delta K(s,t)e^{sx}n!L_n^{-1/2}\big(-(1+s)x\big)ds. \tag{4.4}$$

By modifying the path of integration as in Theorem 4.1, this is also

$$\begin{aligned} p_{2n}(x,t) &= 2x^{-n-\frac{1}{2}} \int_0^\infty e^{-x\cosh^2 y} q(y,t) \cos\Big(\frac{\pi y}{2t}\Big) n! L_n^{-1/2}\big(x\sinh^2 y\big)\, dy \\ &= x^{-n-\frac{1}{2}} \int_{-\infty}^\infty e^{-x\cosh^2 y} q(y,t) e^{\frac{i\pi y}{2t}} n! L_n^{-1/2}\big(x\sinh^2 y\big)\, dy. \end{aligned}$$

Similarly, if $\mu = 2n+1$ is a positive, odd integer,

$$h^{1,n+1}(s,t) = (1-s)^{-n} h^{2n+1,n+1}(s,t),$$

which implies

$$\begin{aligned} x^{n+1} e^x p_1(x,t) &= \frac{1}{\Gamma(n)} \int_0^x (x-y)^{n-1} e^y y^{n+1} p_{2n+1}(y,t)\, dy \\ p_{2n+1}(x,t) &= x^{-n-1} e^{-x} \frac{d^n}{dx^n} \big[e^x x^{n+1} p_1(x,t)\big] \\ &= x^{-n-1} e^{-x} \int_\Delta K(s,t)\sqrt{1+s} \left(\frac{d^n}{dx^n}[x^{n+\frac{1}{2}} e^{(1+s)x}]\right) ds \\ &= x^{-n-1} \int_\Delta K(s,t) e^{sx} \sqrt{x(1+s)}\, n! L_n^{1/2}\big(-(1+s)x\big)\, ds \qquad (4.5) \\ &= 2x^{-n-1} \int_0^\infty e^{-x\cosh^2 y} q(y,t)\sqrt{x}\sinh y\, \sin\Big(\frac{\pi y}{2t}\Big) \\ &\qquad \times n! L_n^{1/2}\big(x\sinh^2 y\big)\, dy \\ &= -ix^{-n-1} \int_{-\infty}^\infty e^{-x\cosh^2 y} q(y,t)\sqrt{x}\sinh y\, e^{\frac{i\pi y}{2t}} \\ &\qquad \times n! L_n^{1/2}\big(x\sinh^2 y\big)\, dy. \end{aligned}$$

The relationship between Laguerre and Hermite polynomials (see Lebedev, 1972, p. 81),

$$L_n^{-1/2}(x) = \frac{(-1)^n}{2^{2n} n!} H_{2n}(\sqrt{x}) \qquad (4.6)$$

$$L_n^{1/2}(x) = \frac{(-1)^n}{2^{2n+1} n!} \frac{H_{2n+1}(\sqrt{x})}{\sqrt{x}}, \qquad (4.7)$$

makes it possible to rewrite the above expressions for p_{2n} and p_{2n+1} as a single formula

$$p_m(x,t) = (2i)^{-m} x^{-\frac{m+1}{2}} \int_{-\infty}^\infty e^{-x\cosh^2 y} q(y,t) e^{\frac{i\pi y}{2t}} H_m(\sqrt{x}\sinh y)\, dy, \quad m = 0,1,2,\ldots \qquad (4.8)$$

The next theorem shows that the same formula holds when $m \in \mathbb{N}$ is replaced with $\mu \in \mathbb{R}$. The Hermite polynomials are replaced with the Hermite functions (Lebedev, 1972, p. 285):

$$H_\mu(z) \;=\; \frac{2^\mu \Gamma(\frac{1}{2})}{\Gamma\left(\frac{1-\mu}{2}\right)} {}_1F_1\left(-\tfrac{\mu}{2}, \tfrac{1}{2}; z^2\right) + \frac{2^\mu \Gamma(-\frac{1}{2})}{\Gamma\left(-\frac{\mu}{2}\right)} z\, {}_1F_1\left(\tfrac{1-\mu}{2}, \tfrac{3}{2}; z^2\right), \qquad \mu, z \in \mathbb{C}.$$

Hermite functions have the following integral representations (Lebedev, 1972, p. 290):

$$H_\mu(z) \;=\; \frac{1}{\Gamma(-\mu)} \int_0^\infty e^{-\xi^2 - 2\xi z} \xi^{-\mu-1}\, d\xi, \qquad \mathrm{Re}(\mu) < 0 \qquad (4.9)$$

$$= \frac{2^{\mu+1} e^{z^2}}{\sqrt{\pi}} \int_0^\infty e^{-\xi^2} \xi^\mu \cos\left(2z\xi - \frac{\mu\pi}{2}\right) d\xi, \qquad \mathrm{Re}(\mu) > -1. \quad (4.10)$$

Theorem 4.2. *Let $\mu \in \mathbb{R}$, $t, x > 0$ and*

$$q(y,t) \;=\; \frac{e^{\frac{\pi^2}{8t} - \frac{y^2}{2t}}}{\pi\sqrt{2t}} \cosh y\,.$$

The p.d.f. of $1/(2A_t^{(\mu)})$ is $f_\mu(x,t) \;=\; e^{-\mu^2 t/2} p_\mu(x,t)$ with

$$p_\mu(x,t) \;=\; 2^{-\mu} x^{-\frac{\mu+1}{2}} \int_{-\infty}^\infty e^{-x\cosh^2 y} q(y,t) \cos\left(\tfrac{\pi}{2}\left(\tfrac{y}{t} - \mu\right)\right) H_\mu(\sqrt{x}\sinh y)\, dy. \qquad (4.11)$$

The p.d.f. is an entire function of μ. When μ is a non-negative integer, the Hermite function H_μ reduces to a polynomial (see Eqs. (4.4)-(4.7)). Other equivalent expressions are:

(a) If $\mu \neq -1, -3, \ldots$:

$$p_\mu(x,t) \;=\; 2x^{-\frac{\mu+1}{2}} e^{-x} \frac{\Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \int_0^\infty q(y,t) \cos\left(\tfrac{\pi y}{2t}\right) {}_1F_1\left(\tfrac{\mu+1}{2}, \tfrac{1}{2}; -x\sinh^2 y\right) dy \qquad (4.12)$$

$$= 2x^{-\frac{\mu+1}{2}} \frac{\Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \int_0^\infty e^{-x\cosh^2 y} q(y,t) \cos\left(\tfrac{\pi y}{2t}\right) \times {}_1F_1\left(-\tfrac{\mu}{2}, \tfrac{1}{2}; x\sinh^2 y\right) dy. \qquad (4.13)$$

(b) If $\mu \neq -2, -4, \ldots$:

$$p_\mu(x,t) \;=\; 2x^{-\frac{\mu}{2}} e^{-x} \frac{\Gamma\left(\frac{\mu}{2}+1\right)}{\Gamma\left(\frac{3}{2}\right)} \int_0^\infty q(y,t) \sinh y\, \sin\left(\tfrac{\pi y}{2t}\right) \times {}_1F_1\left(\tfrac{\mu}{2}+1, \tfrac{3}{2}; -x\sinh^2 y\right) dy \qquad (4.14)$$

$$= 2x^{-\frac{\mu}{2}} \frac{\Gamma\left(\frac{\mu}{2}+1\right)}{\Gamma\left(\frac{3}{2}\right)} \int_0^\infty e^{-x\cosh^2 y} q(y,t) \sinh y\, \sin\left(\tfrac{\pi y}{2t}\right) \times {}_1F_1\left(\tfrac{1-\mu}{2}, \tfrac{3}{2}; x\sinh^2 y\right) dy. \qquad (4.15)$$

Proof. Let $t, x > 0$, $\mu < n$, $n \in \mathbb{N}$. From Theorem 3.1,

$$h^{\mu,(n+\mu)/2}(s,t) \;=\; (1-s)^{(\mu-n)/2} h^{n,(n+\mu)/2}(s,t),$$

and thus

$$p_\mu(x,t) \;=\; \frac{x^{-\frac{n+\mu}{2}} e^{-x}}{\Gamma\left(\frac{n-\mu}{2}\right)} \int_0^x (x-y)^{\frac{n-\mu}{2}-1} y^{\frac{n+\mu}{2}} e^y p_n(y,t)\, dy. \tag{4.16}$$

Define $\tilde{p}_{x,t}(\mu)$ as the right-hand side of (4.16), for $\{\mathrm{Re}(\mu) < n\}$. From the expression for p_n in (4.8), $\tilde{p}_{x,t}(\mu)$ is analytic in $\{\mathrm{Re}(\mu) < n\}$, for every $n \in \mathbb{N}$, and is hence an entire function of μ; $\tilde{p}_{x,t}(\mu)$ coincides with $p_\mu(x,t)$ for $\mu \in \mathbb{R}$.

Rewrite (4.2) and (4.3) as

$$p_0(y,t) \;=\; y^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-y\cosh^2 u} e^{\frac{i\pi u}{2t}} q(u,t)\, du$$

$$p_1(y,t) \;=\; -i\, y^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-y\cosh^2 u} \sinh u\; e^{\frac{i\pi u}{2t}}\, q(u,t)\, du.$$

If $n = 0$ and $-1 < \mu < 0$, (4.16) yields

$$\begin{aligned}
& x^{\frac{\mu}{2}} e^x p_\mu(x,t) \\
&\quad = \frac{1}{\Gamma\left(-\frac{\mu}{2}\right)} \int_0^x (x-y)^{-\frac{\mu}{2}-1} y^{\frac{\mu-1}{2}} e^y \int_{-\infty}^{\infty} e^{-y\cosh^2 u} e^{\frac{i\pi u}{2t}} q(u,t)\, du\, dy \\
&\quad = \frac{1}{\Gamma\left(-\frac{\mu}{2}\right)} \int_{-\infty}^{\infty} e^{\frac{i\pi u}{2t}} q(u,t) \int_0^x (x-y)^{-\frac{\mu}{2}-1} y^{\frac{\mu-1}{2}} e^{-y\sinh^2 u}\, dy\, du \\
&\quad = \frac{x^{-\frac{1}{2}}}{\Gamma\left(-\frac{\mu}{2}\right)} \int_{-\infty}^{\infty} e^{\frac{i\pi u}{2t}} q(u,t) \int_0^1 (1-v)^{-\frac{\mu}{2}-1} v^{\frac{\mu-1}{2}} e^{-vx\sinh^2 u}\, dv\, du \\
&\quad = \frac{x^{-\frac{1}{2}}\Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \int_{-\infty}^{\infty} e^{\frac{i\pi u}{2t}} q(u,t)\, {}_1F_1\left(\tfrac{\mu+1}{2}, \tfrac{1}{2}; -x\sinh^2 u\right) du \qquad (4.17) \\
&\quad = \frac{x^{-\frac{1}{2}}\Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \int_{-\infty}^{\infty} e^{\frac{i\pi u}{2t}} q(u,t) e^{-x\sinh^2 u}\, {}_1F_1\left(-\tfrac{\mu}{2}, \tfrac{1}{2}; x\sinh^2 u\right) du. \qquad (4.18)
\end{aligned}$$

The interchange of integrals is justified by absolute convergence of the double integral; the last two equalities result from the usual properties of confluent hypergeometric function (Lebedev, 1972, pp. 266-267):

$$\begin{aligned}
{}_1F_1(a,c;z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{c-a-1}\, dt, & \mathrm{Re}(c) > \mathrm{Re}(a) > 0 \\
{}_1F_1(a,c;z) &= e^z\, {}_1F_1(c-a, c; -z), & -c \notin \mathbb{N}.
\end{aligned}$$

The same can be repeated based on p_1 instead of p_0. If $n = 0$ and $-2 < \mu < 1$, (4.16) gives

$$\begin{aligned}
&x^{\frac{\mu+1}{2}} e^x p_\mu(x,t) \\
&\quad = \frac{1}{\Gamma\left(\frac{1-\mu}{2}\right)} \int_0^x (x-y)^{-\frac{\mu+1}{2}} y^{\frac{\mu+1}{2}} e^y p_1(y,t)\, dy \\
&\quad = \frac{-i}{\Gamma\left(\frac{1-\mu}{2}\right)} \int_0^x (x-y)^{-\frac{\mu+1}{2}} y^{\frac{\mu}{2}} e^y \int_{-\infty}^{\infty} e^{-y\cosh^2 u} \sinh u \; e^{\frac{i\pi u}{2t}}\, q(u,t)\, du\, dy \\
&\quad = \frac{-i}{\Gamma\left(\frac{1-\mu}{2}\right)} \int_{-\infty}^{\infty} \sinh u \; e^{\frac{i\pi u}{2t}}\, q(u,t) \int_0^x (x-y)^{-\frac{\mu+1}{2}} y^{\frac{\mu}{2}} e^{-y\sinh^2 u}\, dy\, du \\
&\quad = \frac{-i\, x^{\frac{1}{2}}}{\Gamma\left(\frac{1-\mu}{2}\right)} \int_{-\infty}^{\infty} \sinh u \; e^{\frac{i\pi u}{2t}}\, q(u,t) \int_0^1 (1-v)^{-\frac{\mu+1}{2}} v^{\frac{\mu}{2}} e^{-vx\sinh^2 u}\, dv\, du \\
&\quad = \frac{-i\, x^{\frac{1}{2}}\, \Gamma\left(\frac{\mu}{2}+1\right)}{\Gamma\left(\frac{3}{2}\right)} \int_{-\infty}^{\infty} \sinh u \; e^{\frac{i\pi u}{2t}}\, q(u,t)\, {}_1F_1\left(\tfrac{\mu}{2}+1, \tfrac{3}{2}; -x\sinh^2 u\right) du \qquad (4.19) \\
&\quad = \frac{-i\, x^{\frac{1}{2}}\, \Gamma\left(\frac{\mu}{2}+1\right)}{\Gamma\left(\frac{3}{2}\right)} \int_{-\infty}^{\infty} \sinh u \; e^{\frac{i\pi u}{2t}}\, q(u,t) e^{-x\sinh^2 u}\, {}_1F_1\left(\tfrac{1-\mu}{2}, \tfrac{3}{2}; x\sinh^2 u\right) du. \qquad (4.20)
\end{aligned}$$

Eqs. (4.18) and (4.20) are two different expressions for the same function, if $-1 < \mu < 0$. The p.d.f. may thus also be expressed as a convex combination of these two expressions. Apply the following weights to (4.18) and (4.20), respectively:

$$\frac{1}{2}(1+e^{-i\pi\mu}), \qquad \frac{1}{2}(1-e^{-i\pi\mu}).$$

We get

$$x^{\frac{\mu+1}{2}} p_\mu(x,t) \;=\; \int_{-\infty}^{\infty} e^{-x\cosh^2 u}\, e^{\frac{i\pi u}{2t}}\, q(u,t)\phi_\mu(z^2)\, du, \quad -1 < \mu < 0, \qquad (4.21)$$

where $z = \sqrt{x}\sinh u$ and, recalling the formula $\Gamma(w) = \dfrac{\pi}{\sin(\pi w)\Gamma(1-w)}$,

$$\begin{aligned}
\phi_\mu(z^2) \;&=\; \frac{1}{2}(1+e^{-i\pi\mu}) \frac{\Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\, {}_1F_1\left(-\tfrac{\mu}{2}, \tfrac{1}{2}; z^2\right) \\
&\quad - \frac{i}{2}(1-e^{-i\pi\mu}) \frac{\Gamma\left(\frac{\mu}{2}+1\right)}{\Gamma\left(\frac{3}{2}\right)}\, z\, {}_1F_1\left(\tfrac{1-\mu}{2}, \tfrac{3}{2}; z^2\right) \\
&= \frac{1+e^{-i\pi\mu}}{2\sin\left(\pi\left(\frac{\mu+1}{2}\right)\right)} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-\mu}{2}\right)}\, {}_1F_1\left(-\tfrac{\mu}{2}, \tfrac{1}{2}; z^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{i(1-e^{-i\pi\mu})}{2\sin\left(\pi\left(\frac{\mu}{2}+1\right)\right)}\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{\mu}{2}\right)}z\,{}_1F_1\left(\tfrac{1-\mu}{2},\tfrac{3}{2};z^2\right)\\
&=\frac{1+e^{-i\pi\mu}}{e^{\frac{i\pi\mu}{2}}+e^{-\frac{i\pi\mu}{2}}}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-\mu}{2}\right)}\,{}_1F_1\left(-\tfrac{\mu}{2},\tfrac{1}{2};z^2\right)\\
&+\frac{1-e^{-i\pi\mu}}{e^{\frac{i\pi\mu}{2}}-e^{-\frac{i\pi\mu}{2}}}\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{\mu}{2}\right)}z\,{}_1F_1\left(\tfrac{1-\mu}{2},\tfrac{3}{2};z^2\right)\\
&=e^{-\frac{i\pi\mu}{2}}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-\mu}{2}\right)}\,{}_1F_1\left(-\tfrac{\mu}{2},\tfrac{1}{2};z^2\right)+e^{-\frac{i\pi\mu}{2}}\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{\mu}{2}\right)}z\,{}_1F_1\left(\tfrac{1-\mu}{2},\tfrac{3}{2};z^2\right)\\
&=2^{-\mu}e^{-i\pi\mu/2}H_\mu(z).
\end{aligned}
$$

This proves (4.11) for $-1<\mu<0$. But the right-hand side of (4.11) is an entire function of μ, as can be seen from the integral representations of H_μ (Eqs. (4.9)-(4.10)). Eq. (4.11) holds for all $\mu\in\mathbb{R}$ by analytic continuation, since $p_\mu(x,t)$ coincides with the entire function $\tilde{p}_{x,t}(\mu)$ for $\mu\in\mathbb{R}$.

Formulas (4.12)-(4.13) are obtained by noting that expressions (4.17)-(4.18) are entire functions of μ, when divided by $\Gamma((\mu+1)/2)$. The same argument applies to formulas (4.19)-(4.20), yielding (4.14)-(4.15). □

Remarks. **1**. The simplifications which occur when μ is a non-negative integer are more or less apparent in Theorem 5 of Dufresne (1997), which includes the identity

$$
\mathsf{E}\,e^{-s/2A^{(\mu)}_{T_\lambda}} \;=\; \mathsf{E}\left(\prod_{k=1}^{\infty}\frac{1}{1+sU_1\cdots U_k}\right)^{\mu}
$$

where $\mu>0$, T_λ is as in Theorem 2.1, and $\{U_k;\,k\geq 1\}$ are independent variables with the distribution of the ratio of two independent beta variables (as before $\beta_\mu=-\frac{\mu}{2}+\frac{1}{2}\sqrt{2\lambda+\mu^2}$):

$$
U\;\stackrel{\mathcal{L}}{=}\;\frac{B_1}{B_2},\qquad B_1\sim\mathrm{Beta}(\beta_\mu,\mu),\qquad B_2\sim\mathrm{Beta}(1+\beta_\mu,\mu).
$$

2. Another proof can be given for the recursion (Comtet *et al.*, 1998, p.269)

$$
\frac{d}{dt}\mathsf{E}\,e^{-2sA^{(\mu)}_t}\;=\;-2se^{2(\mu+1)t}\mathsf{E}\,e^{-2sA^{(\mu+2)}_t}.
$$

Define $Y^{(\mu)}_t$ as in (2.7). Ito's formula yields

$$
\frac{d}{dt}\mathsf{E}\,e^{-2sY^{(\mu)}_t}\;=\;-2s\mathsf{E}[1+(\mu+1)2Y^{(\mu)}_t-s(2Y^{(\mu)}_t)^2]e^{-2sY^{(\mu)}_t},
$$

From Theorem 3.1,

$$
h^{\mu,\mu+1}(s,t)\;=\;(1-s)^{-1}h^{\mu+2,\mu+1}(s,t)
$$

and thus

$$x^{\mu+1}e^x p_\mu(x,t) \;=\; \int_0^x y^{\mu+1}e^y p_{\mu+2}(y,t)\,dy$$

or

$$p_{\mu+2}(x,t) \;=\; x^{-\mu-1}e^{-x}\frac{d}{dx}\left[x^{\mu+1}e^x p_\mu(x,t)\right]. \tag{4.22}$$

Hence, for $\mu \in \mathbb{R}$, $\mathrm{Re}(s) \le 0$,

$$\begin{aligned}
\mathsf{E}\, e^{-2sY_t^{(\mu+2)}} &= e^{-(\mu+2)^2t/2}\int_0^\infty e^{-s/x}p_{\mu+2}(x,t)\,dx \\
&= -e^{-(\mu+2)^2t/2}\int_0^\infty x^{\mu+1}e^x p_\mu(x,t)d(e^{-s/x-x}x^{-\mu-1}) \\
&= e^{-(\mu+2)^2t/2}\int_0^\infty \left(1+\frac{\mu+1}{x}-\frac{s}{x^2}\right)e^{-s/x}p_\mu(x,t)\,dx \\
&= e^{-2(\mu+1)t}\mathsf{E}[1+(\mu+1)2Y_t^{(\mu)}-s(2Y_t^{(\mu)})^2]e^{-2sY_t^{(\mu)}}. \qquad \square
\end{aligned}$$

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