Scand. Actuarial J. 1990: 39–79

The Distribution of a Perpetuity, with Applications to Risk Theory and Pension Funding

By Daniel Dufresne, Université de Montréal

Abstract

If V_k is the discount factor for the kth period, then $Z=\Sigma_{k\geq 1} V_1 \ldots V_k C_k$ is the discounted value of a perpetuity paying C_k at time k. In some cases Z is also the limiting distribution of $S_t=V_t(S_{t-1}+C_{t-1})$. This paper

(1) reviews the literature concerning Z and $\{S_t\}$;

(2) considers continuous-time counterparts of Z and S, at the same time deriving the distribution of $\int \exp(-\gamma t-\sigma W_t)1_{(0,\infty)}(t)dt$ when W is Brownian motion;

(3) gives applications to risk theory and pension funding.

Key words: random rates of return, stochastic difference equations, discounted sums of random variables.

1. Introduction

Suppose R_k is the rate of return during the period $(k-1, k)$, and assume $\{R_k, k\geq 1\}$ is i.i.d. The accumulated value at time t of payments of one unit made at times 0, 1, ..., $t-1$ is then

$$S_t=(1+R_t)+(1+R_t)(1+R_{t-1})+\ldots+(1+R_t)\ldots(1+R_1) \tag{1.1}$$

$$=(1+R_t)(S_{t-1}+1), \quad S_0=0. \tag{1.2}$$

The independence of $\{R_t\}$ greatly simplifies the study of $\{S_t\}$. The latter is then a Markov process, the moments of which can be calculated recursively:

$$ES_t^k=E(1+R_t)^k\left(1+\sum_{j=1}^{k} ES_{t-1}^j\right).$$

Given a particular distribution for R_t, the distributions of S_1, S_2, ... can (at least in theory) be calculated recursively using convolutions (see (1.2)).

Now turn to discounted values. If $P(R_k=-1)=0$, then the discounted value, at time 0, of payments of one unit made at times 1, 2, ..., t is

$$Z_t=V_1+V_1 V_2+\ldots+V_1\ldots V_t$$

with $V_k=1/(1+R_k)$. Observe that payments are made at the end of the year, rather than at the beginning of the year as in Eq. (1.1). The process $\{Z_t\}$ has a structure very different from that of $\{S_t\}$. The sequence of discounted values is not a Markov process; its distributions appear to be harder to derive; even moments are difficult to calculate (e.g. see Boyle, 1976). One way of getting round these problems is to define a new sequence $\{B_t\}$ as follows:

$$B_t = V_t + V_t V_{t-1} + \ldots + V_t \ldots V_1$$
$$\stackrel{d}{=} Z_t$$

B_t and Z_t are the same sum of products, except that the order of the discount factors $V_1, \ldots, V_t$ has been reversed. What matters is that B_t has the same distribution as Z_t (written "$B_t \stackrel{d}{=} Z_t$") and that $\{B_t\}$ has the same structure as $\{S_t\}$:

$$B_t = V_t(B_{t-1}+1), t \geqslant 1, \quad B_0 = 0.$$

The moments and one-dimensional distributions of $\{Z_t\}$ can therefore be found recursively using the techniques described above for $\{S_t\}$.

This paper will deal with processes such as $\{S_t\}$ and $\{Z_t\}$. It will more generally be supposed that the payments are random, that is to say we consider

$$Z_t = \sum_{k=1}^{t} V_1 \ldots V_k C_k \tag{1.3}$$

$$S_t = (1+R_t)(S_{t-1}+C_{t-1}). \tag{1.4}$$

There is a sizeable literature related to processes satisfying either (1.3) or (1.4). A comprehensive list of references is given in Vervaat (1979). I will deal mostly with the case where $\{V_k\}$ and $\{C_k\}$ (respectively $\{R_k\}$ and $\{C_k\}$) are independent i.i.d. sequences. A number of results will nevertheless be shown to apply in more general situations, using the approach of Brandt (1986).

Section 2 looks at the moments of $\{S_t\}$ and $\{Z_t\}$, with a brief allusion to life annuities. Section 3 gives general conditions for the existence of the perpetuity $Z=\lim_{t\to\infty} Z_t$. Four explicit examples of distributions of Z are given (one of them is new).

Section 4 describes continuous-time counterparts of $\{S_t\}$ and $\{Z_t\}$. Weak convergence proofs are provided. The distribution of the continuous perpetuity

$$Z = \int_0^\infty \exp-(\gamma t + \sigma W_t)\, dt$$

is derived, when W is standard Brownian motion. This section is a continuation of Dufresne (1989).

In Section 5 the results of Section 3 are applied to the determination of the distribution of discounted future claims

$$\sum_{k\geq 1} e^{-\delta T_k} C_k$$

in a classical risk theoretic setting. Diffusion approximations of the surplus process are also discussed. Finally (Section 6), the same ideas are used in examining the evolution of a pension fund over time, assuming a simple "proportional feedback" adjustment of contributions.

2. Moments of S_t and Z_t

2.1. A number of authors have studied the moments of annuities-certain and life annuities when rates of return are random (see Dufresne, 1989, for references). This section is only concerned with the case of i.i.d. rates of return. The idea is to derive the moments by applying the classical theory of difference equations. A good introduction to this theory is Goldberg (1986).

2.2. Assume $\{U_k\}$ and $\{C_k\}$ are two independent i.i.d. sequences and define

$$S_t = U_t(S_{t-1}+C_{t-1}), t\geq 1, \quad S_0=0.$$

S_t represents the accumulated value, at time t, of payments $C_0, \ldots, C_{t-1}$, when rates of returns are $R_1=U_1-1, \ldots, R_t=U_t-1$.

Letting $u_j=EU_1^j$ and $c_j=EC_0^j$,

$$ES_t^m = u_m\left(c_m+\sum_{j=1}^{m}\binom{m}{j}c_{m-j}ES_{t-1}^j\right),$$

that is to say

$$ES_t^m - u_m ES_{t-1}^m = u_m c_m + \sum_{j=1}^{m-1}\binom{m}{j}u_m c_{m-j}ES_{t-1}^j. \tag{2.1}$$

Thus $\varphi_t=ES_t^m$ satisfies a (non-homogeneous) first-order linear difference equation. The solution of the corresponding homogeneous equation

$$\varphi_t - u_m\varphi_{t-1}=0$$

is $\varphi_t=(\text{constant})\cdot u_m^t$. In order to find a particular solution to Eq. (2.1), proceed by induction. When $m=1$, the complete difference equation reduces to

$$ES_t - u_1 ES_{t-1} = u_1 c_1.$$

A particular solution is the constant $u_1 c_1/(1-u_1)$, unless $u_1=1$, and the complete solution, given $ES_0=0$, is

$$ES_t = (u_1 c_1 u_1^t - u_1 c_1)/(u_1 - 1) \tag{2.2}$$
$$= (cst)+(cst)\cdot u_1^t.$$

The case $u_1=1$ will be dealt with later; for now suppose $u_1 \neq 1$. Substituting ES_t into the right hand side of Eq. (2.1), the difference equation satisfied by $\{ES_t^2, t \geq 1\}$ becomes

$$ES_t^2 - u_2 ES_{t-1}^2 = u_2 c_2 + 2u_2 c_1 ES_{t-1}$$
$$= (cst)+(cst)\cdot u_1^t.$$

A particular solution to this equation is of the same form as its right hand side, unless $u_2=u_1$ or $u_2=1$; suppose this is not the case. Then

$$ES_t^2 = (cst)+(cst)\cdot u_1^t + (cst)\cdot u_2^t.$$

More generally, suppose

$$u_i \neq u_j, \quad 0 \leq i < j \leq m \tag{2.3}$$

(observe that $u_0 = EU^0 = 1$). Then the right hand side of Eq. (2.1) is of the form

$$(cst)+(cst)\cdot u_1^t + \ldots + (cst)\cdot u_{m-1}^t.$$

In consequence

$$ES_t^m = d_{m0} + \sum_{j=1}^{m} d_{mj} u_j^t, \tag{2.4}$$

where $\{d_{mj}, 0 \leq j \leq m\}$ are constants.

No explicit expression has been found for d_{mj}. Observe that such a closed form expression does exists (when payments are constant) for the continuous version of S_t described in paragraph 4.3.

One way of finding the parameters $\{d_{mj}\}$ is to substitute (2.4) into (2.1) to get

$$\sum_{j=0}^{m-1} d_{mj}\left(1-\frac{u_m}{u_j}\right) u_j^t = \sum_{i=0}^{m-1} \binom{m}{i} u_m c_{m-i} \sum_{k=0}^{i} \frac{d_{ik}}{u_k} u_k^t$$

and then reverse the order of summation on the r.h.s.. By looking at the coefficients of $u_o^t, \ldots, u_{m-1}^t$, we obtain

$$d_{mj} = \frac{u_m}{u_j - u_m} \sum_{i=j}^{m-1} \binom{m}{i} c_{m-i} d_{ij}, \; 0 \leq j \leq m-1. \tag{2.5}$$

Thus $\{d_{ij},\ 0 \leq j \leq i \leq m-1\}$ yield $\{d_{mj},\ 0 \leq j \leq m-1\}$; d_{mm} can then be found by noting that $ES_0^m=0$ implies (Eq. (2.4))

$$d_{mm} = -\sum_{j=0}^{m-1} d_{mj}. \tag{2.6}$$

For example, $d_{00}=1$ implies $d_{10}=u_1 c_1/(1-u_1)$ by Eq. (2.5) and then Eq. (2.6) gives $d_{11}=u_1 c_1/(u_1-1)$; this corresponds to Eq. (2.2). Similarly,

$$ES_t^2 = d_{20}+d_{21} u_1^t+d_{22} u_2^t$$

where

$$d_{20} = \frac{u_2 c_2}{1-u_2} + \frac{2u_1 u_2 c_1^2}{(1-u_2)(1-u_1)}$$

$$d_{21} = \frac{2u_1 u_2 c_1^2}{(u_1-u_2)(u_1-1)}$$

$$d_{22} = -(d_{20}+d_{21})$$

and so on.

When conditions (2.3) are not satisfied the calculations are slightly different. For example, suppose $U \sim \log \mathbf{N}(\mu, \sigma^2)$ with $\mu=-3\sigma^2/2$, so that $u_1 \neq 1$ but $u_1=u_2$. Then Eq. (2.2) holds as before but when $m=2$ the solution of Eq. (2.1) becomes

$$ES_t^2 = e_{20}+e_{21} u_1^t+e_{22} t u_1^t$$

where e_{20}, e_{21} and e_{22} are constants. Substituting this expression into Eq. (2.1), we get

$$(1-u_1) e_{20}+e_{22}u_1^t = u_1 c_2+2u_1 c_1 d_{10}+2u_1 c_1 d_{11} u_1^{t-1}$$

or

$$e_{20} = \frac{u_1 c_2}{1-u_1} + \frac{2u_1^2 c_1^2}{(1-u_1)^2}$$

$$e_{22} = \frac{2u_1 c_1^2}{u_1-1}.$$

Finally $ES_0^2=0$ implies $e_{21}=-e_{20}$. Higher moments are also different. In this case they would be

$$ES_t^m = e_{m0}+e_{m1}u_1^t+e_{m2}tu_1^t+e_{m3}u_3^t+\ldots+e_{mm}u_m^t.$$

Remark 2.2.1. When $U \geq 0$ and $EU>1$ conditions (2.3) are always satisfied, since for $m \geq 2$

$$(EU^m)^{1/m} \geqslant (EU^{m-1})^{1/(m-1)}$$
$$\Rightarrow EU^m \geqslant (EU^{m-1})^{1+1/(m-1)} > EU^{m-1}.$$

The first inequality is due to the fact that $\|X\|_t=(E|X|^t)^{1/t}$ is a non decreasing function of $t>0$; see Feller (1971, p. 155) or Loève (1977, p. 158).

2.3. Suppose $\{C_k\}$ and $\{V_k\}$ are two independent i.i.d. sequences, and define

$$Z_t = \sum_{j=1}^{t} C_j V_1 \dots V_j. \tag{2.7}$$

Z_t may represent the discounted value, at time 0, of payments $C_1, \dots, C_t$, when rates of return are $R_1=V_1^{-1}-1, \dots, R_t=V_t^{-1}-1$.

We have

$$Z_t \overset{d}{=} C_t V_t + C_{t-1} V_t V_{t-1} + \dots + C_1 V_t \dots V_1$$
$$\overset{d}{=} C_{t-1} V_t + C_{t-2} V_t V_{t-1} + \dots + C_0 V_t \dots V_1. \tag{2.8}$$

Calling B_t the r.h.s. of this equation ($B_0=0$), we get

$$B_t = V_t(B_{t-1}+C_{t-1}), \quad t \geqslant 1. \tag{2.9}$$

The process $\{B_t\}$ has therefore the same structure as $\{S_t\}$, with $\{V_t\}$ playing the role of $\{U_t\}$. We conclude that, if the conditions

$$EV^i \neq EV^j, \quad 0 \leqslant i < j \leqslant m \tag{2.10}$$

are satisfied,

$$EZ_t^m = EB_t^m$$
$$= d_{m0} + \sum_{j=1}^{m} d_{mj} v_j^t.$$

Here $v_j=EV^j$ and the constants $\{d_{mj}\}$ can be found recursively from

$$d_{mj} = \frac{v_m}{v_j - v_m} \sum_{i=j}^{m-1} \binom{m}{i} c_{m-i} d_{ij}.$$

2.4. Let K take values in **N** and be independent of $\{C_k\}$ and $\{V_k\}$. In what follows K represents the curtate future lifetime of an individual. The r.v.

$$\tilde{A} = V_1 \dots V_{K+1}$$

stands for the discounted value of an insurance of one unit payable to the individual at the end of the year of death. If

$$A(i)=\sum_{k\geq 0}(1+i)^{-k-1}P(K=k)$$

is the usual actuarial value of the insurance at constant rate of interest i, then clearly

$$\begin{aligned}E\tilde{A}^m &= E\,E(\tilde{A}^m|K)\\ &= E\,v_m^{K+1}\\ &= A(v_m^{-1}-1).\end{aligned}$$

Furthermore, consider a "life annuity" providing a random payment C_j at time j, while the individual is alive. The discounted value of this life annuity is

$$Z=\sum_{j=1}^{K}C_jV_1\ldots V_j.$$

If conditions (2.10) are satisfied

$$\begin{aligned}EZ^m &= E\,E(Z^m|K)\\ &= E\left[d_{m0}+\sum_{j=1}^{m}d_{mj}v_j^K\right]\\ &= d_{m0}+\sum_{j=1}^{m}(d_{mj}/v_j)\,A(v_j^{-1}-1).\end{aligned}$$

3. The distribution of discrete perpetuities

3.1. The distribution of Z_t (Eq. (2.7)) is in general difficult to derive. In a number of situations, however, it is possible to calculate explicitly the limit distribution of Z_t as $t\to\infty$. First, paragraph 3.2 gives sufficient conditions for the existence of such a distribution, without assuming $\{(C_k, V_k), k\geq 1\}$ to be i.i.d. Secondly, paragraph 3.3 describes a useful technique for calculating the limit distribution of Z_t when $\{C_k\}$ and $\{V_k\}$ are independent i.i.d. sequences. Then three known examples of limit distributions are given (paragraph 3.4). Finally, a new example of limit distribution is detailed in paragraph 3.5.

This section relies heavily on Vervaat (1979) and Brandt (1986).

3.2. The existence of $\lim_{t\to\infty}Z_t$ is easily ascertained if $P(V_1=0)>0$:

Proposition 3.2.1. *If* $\{V_k\}$ *is stationary and ergodic with* $P(V_1=0)>0$, *then* $\lim_{t\to\infty}Z_t$ *exists a.s.*

Proof. $P(V_k=0$ infinitely often$)=1$ means that the limit of (2.7) is a.s. constituted of a finite member of terms. □

A useful device for dealing with more general cases is the root test for the convergence of series: if

$$\limsup_{k\to\infty} |C_k V_1 \dots V_k|^{1/k} < 1 \quad \text{a.s.} \tag{3.1}$$

then $\lim_{t\to\infty} Z_t \in \mathbf{R}$ a.s., i.e. the limit distribution exists.

Proposition 3.2.2. *If* $\{(C_k, V_k), k\geq 1\}$ *is stationary and ergodic, and if*

$$E(\log|C_1|)_+ < \infty, E\log|V_1| < 0$$

then Z_t *converges absolutely a.s.*

Proof. The ergodicity assumption entails

$$\limsup_{k\to\infty} k^{-1}(\log|C_k| + \sum_{j=1}^{k} \log|V_j|) < 0$$

which implies (3.1). □

Propositions 3.2.1 and 3.2.2 are extensions, due to Brandt (1986), of Theorem 1.6 of Vervaat (1979). Vervaat also discusses necessary conditions for convergence.

When Z_t is interpreted as a discounted value, Proposition 3.2.2 says that it will have a finite limit as $t\to\infty$ if the distribution of the payments is not too dispersed, and if the geometric rate of return $(-\log V_1)$ is on average positive. Observe first that $(\log|C|)_+ \leq |C|$ and thus $E|C|<\infty$ implies $E(\log|C|)_+<\infty$. Secondly, by Jensen's inequality

$$EV_1 < 1 \text{ implies } E\log V_1 < 0.$$

Hence the value of a perpetuity is a.s. finite if payments have a finite mean and if the mean discount factor is smaller than 1.

In actuarial applications it may very well turn out that neither $\{C_k\}$ nor $\{V_k\}$ is stationary. The following proposition deals with one such situation.

Suppose $\{C_k\}$ and $\{\log V_k\}$ are random walks

$$C_k = c + C_{k-1} + e_k$$

$$\log V_k = a + \log V_{k-1} + f_k$$

where $\{e_k\}$ and $\{f_k\}$ are two zero-mean i.i.d. sequences. The variances need not be finite, nor do the sequences have to be mutually independent.

Proposition 3.2.3. *If* $a<0$ *then* Z_t *converges a.s.*

Proof. $\log V_k \to -\infty$ a.s. as $k\to\infty$. This implies $k^{-1}\Sigma_{j=1}^{k} \log V_j \to -\infty$ and thus $|V_1 \ldots V_k|^{1/k} \to 0$ a.s. Furthermore, if $c \neq 0$

$$k^{-1}\log|C_k| = k^{-1}\log|ck+C_0+\sum_{j=1}^{k} e_j|$$

$$= k^{-1}\log|ck|+k^{-1}\log\left|1+(ck)^{-1}\left(C_0+\sum_{j=1}^{k} e_j\right)\right| \to 0 \quad \text{a.s.}$$

When $c=0$, $k^{-1}\log|C_k| \leq k^{-1}|C_0+\Sigma_{j=1}^{k} e_j| \to 0$ a.s. □

A perpetuity can thus be a.s. finite even though $C_k \to \infty$ a.s. and Var $C_k = \infty$.

3.3. The remainder of Section 3 is concerned solely with the case of independent i.i.d. sequences $\{C_k\}$ and $\{V_k\}$. The following simple result yields an efficient technique for finding the limit distribution of Z_t.

Proposition 3.3.1. (Vervaat, 1979). *Suppose $\{C_k\}$ and $\{V_k\}$ are independent i.i.d. sequences. If $Z_t \xrightarrow{d} Z$ as $t \to \infty$, then Z satisfies the stochastic equation*

$$Z \stackrel{d}{=} V_1(Z+C_1),\ V_1, Z \text{ and } C_1 \text{ independent.} \tag{3.2}$$

Proof. From Eq. (2.8) the left hand side of (2.9) converges weakly to Z, while its right hand side converges weakly to $V_1(Z+C_1)$. □

3.4. Below are three explicit distributions for Z which have appeared in the literature. Other examples can be found in Vervaat (1972, 1979).

Example 3.4.1. $C \sim \mathbf{exp}(1)$, $V \sim \boldsymbol{\beta}^{(1)}(a, 1)$, i.e.

$$f_V(v) = av^{a-1}1_{(0,1)}(v).$$

This amounts to saying that the geometric rate of return $-\log V$ has an exponential distribution with mean $1/a$. This example goes back to Takacs (1954), though the following derivation is taken from Vervaat (1979).

Let $\eta(s)=Ee^{isZ}$ and $\varphi(s)=Ee^{isC}$. Then (3.2) implies

$$\eta(s) = \int_0^1 av^{a-1}\eta(sv)\,\varphi(sv)\,dv$$

$$= as^{-a}\int_0^s u^{a-1}\eta(u)\,\varphi(u)\,du.$$

Multiplying both sides by s^a and differentiating yields

$$s^a\eta' + as^{a-1}\eta = as^{a-1}\eta\varphi$$

or

$$\eta'/\eta = as^{-1}(\varphi-1).$$

Thus

$$\eta(s) = \exp a \int_0^s (\varphi(u)-1)/u\, du. \tag{3.3}$$

Replacing $\varphi(u)$ by $(1-iu)^{-1}$ we obtain

$$\eta(s) = (1-is)^{-a}$$

or $Z \sim \boldsymbol{\Gamma}(a, 1)$. Finally, multiplying all the C's by a constant m is the same as multiplying Z by m, and so $V \sim \boldsymbol{\beta}^{(1)}(a, 1)$, $C \sim \mathbf{exp}(m)$ imply $Z \sim \boldsymbol{\Gamma}(a, m)$, i.e.

$$f_Z(x) = \Gamma(a)^{-1} m^{-a} x^{a-1} e^{-x/m} 1_{(0,\infty)}(x).$$

This example will be continued in Section 5 (Example 5.1.2.).

Example 3.4.2. (Lassner, 1974; Vervaat, 1979). Suppose $P(V=1)=p=1-P(V=0)=1-q$, $0<p<1$. Then

$$\begin{aligned} Z_t &= \sum_{j=1}^{k} C_j \text{ with probability } p^k q,\ 0 \leq k \leq t-1 \\ &= \sum_{j=1}^{t} C_j \text{ with probability } p^t. \end{aligned}$$

Thus, if $\varphi(s) = Ee^{isC}$

$$Ee^{isZ_t} = q(1-p^t\varphi^t)/(1-p\varphi)+p^t\varphi^t.$$

When $t \to \infty$, this expression converges to

$$Ee^{isZ} = q/(1-p\varphi),$$

i.e. Z has a compound geometric distribution.

Example 3.4.3. (Chamayou and Letac, 1991). Suppose $C \equiv 1$ and $V \sim \boldsymbol{\beta}^{(2)}(a, b)$, i.e.

$$f_V(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1+x)^{-a-b} 1_{(0,\infty)}(x), \quad a, b > 0.$$

This is the "beta distribution of the second kind", also known as generalized Pareto, generalized F, or transformed beta distribution (since $W \sim \boldsymbol{\beta}^{(1)}(b, a) \Rightarrow W^{-1}-1 \sim \boldsymbol{\beta}^{(2)}(a, b)$). It will be shown that if $a<b$ then

$$Z = \sum_{j=1}^{\infty} V_1 \ldots V_j \sim \boldsymbol{\beta}^{(2)}(a, b-a).$$

Let

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt,$$

$$\Psi(s) = \Gamma'(s)/\Gamma(s)$$
$$= -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+s} \right) \quad (3.4)$$

be the gamma and digamma functions, respectively, where γ is Euler's constant (see Lebedev, 1972, p. 7).

First, let us show that $E \log V < 0$. Suppose $X \sim \Gamma(a, 1)$ and $Y \sim \Gamma(b, 1)$ are independent random variables. Then (Hogg and Tanis, 1988, p. 279)

$$W = Y/(X+Y) \sim \beta^{(1)}(b, a) \quad (3.5)$$

$$\Rightarrow W^{-1} - 1 = X/Y \sim \beta^{(2)}(a, b) \quad (3.6)$$

and so (by (3.4))

$$E \log V = E \log X - E \log Y$$
$$= \Psi(a) - \Psi(b)$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{n+b} - \frac{1}{n+a} \right).$$
$$< 0.$$

To finish the proof it suffices to show that if $Z \sim \beta^{(2)}(a, b-a)$ and $V \sim \beta^{(1)}(a, b)$, then

$$Z \stackrel{d}{=} V(Z+1), \quad (3.7)$$

V and Z being independent on the r.h.s. of the equation. This will be achieved by using the Mellin transform $M_U(s) = EU^s$, $s \in \mathbf{C}$, U a random variable. Letting X and Y have the same definition as before and performing the required integration yields

$$EX^s = \Gamma(a+s)/\Gamma(a), \quad EY^s = \Gamma(b+s)/\Gamma(b). \quad (3.8)$$

From (3.6)

$$EV^s = EX^s EY^{-s}$$
$$= \Gamma(a+s)\,\Gamma(b-s)/\Gamma(a)\,\Gamma(b).$$

One also finds

$$EZ^s = \Gamma(a+s)\,\Gamma(b-a-s)/\Gamma(a)\,\Gamma(b-a).$$

Calculating the transform of the beta distribution of the first kind and then using (3.5) and (3.6) yields

$$E(Z+1)^s = \Gamma(b)\,\Gamma(b-a-s)/\Gamma(b-a)\,\Gamma(b-s).$$

Therefore $EZ^s = EV^sE(Z+1)^s$, which completes the proof of (3.7). This result will be used in paragraph 4.4 to derive the distribution of a continuous perpetuity.

(N.B. In analysis the Mellin transform of $f(\cdot)$ is usually defined as $\int_0^\infty x^{s-1}f(s)\,ds$. The previous definition has the same properties and is simpler to use.)

3.5. Here is another example of perpetuity with an explicit distribution. Suppose $X \sim \beta^{(1)}(a, 1)$ and $Y \sim \beta^{(1)}(b, 1)$ are independent, and that $V \overset{d}{=} XY$. This assumption can also be expressed as $V \sim \beta^{(1)}(a, 1) \otimes \beta^{(1)}(b, 1)$; it is equivalent to assuming that the geometric rate of return $-\log V$ is the sum of two independent exponential variables, with respective means $1/a$ and $1/b$.

Clearly $E \log V < 0$. In order to find the law of Z, a second-order differential equation will be derived for $\eta(s) = Ee^{isZ}$. Let $\varphi(s) = Ee^{isC}$ and $\zeta(s) = (\varphi(s)-1)/s$. From Proposition 3.3.1

$$Z \overset{d}{=} XY(Z+C), \quad X,\ Y,\ Z \text{ and } C \text{ independent.}$$

Summing over the possible values of X and Y,

$$\eta(s) = \int_0^1 \int_0^1 \eta(sxy)\,\varphi(sxy)\,abx^{a-1}y^{b-1}dydx.$$

With the substitution $u = sx$

$$\eta(s) = s^{-a} \int_0^s \int_0^1 \eta(uy)\,\varphi(uy)\,abu^{a-1}y^{b-1}dydu$$

and

$$s\eta'(s) + a\eta(s) = ab \int_0^1 \eta(sy)\,\varphi(sy)\,y^{b-1}dy$$

$$= ab\,s^{-b} \int_0^s \eta(v)\,\varphi(v)\,v^{b-1}dv.$$

Finally, after differentiating once more,

$$s\eta'' + (1+a+b)\,\eta' - ab\eta\zeta = 0. \tag{3.9}$$

$C \sim \mathbf{exp}(1)$ implies $\zeta(s) = i/(1-is)$ and

$$s(1-is)\eta'' + (1+a+b)(1-is)\eta' - abi\eta = 0.$$

With the change of variables $u = is$ and $\theta(u) = \eta(s) = \eta(-iu)$ the last equation becomes

$$u(1-u)\theta'' + (1+a+b)(1-u)\theta' - ab\theta = 0.$$

This is a differential equation of the "Gauss" or "hypergeometric" type (Lebedev, 1972, pp. 162–163):

$$x(1-x)y'' + [\gamma - (1+\alpha+\beta)x]y' - \alpha\beta y = 0. \tag{3.10}$$

The general solution of this equation can be expressed in terms of the hypergeometric function

$$F(\alpha, \beta, \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \cdot \frac{x^k}{k!}, \qquad (\alpha)_k = \prod_{j=0}^{k-1} (\alpha + j). \tag{3.11}$$

F converges for $|x| < 1$. The general solution of Eq. (3.10) is

$$y(x) = AF(\alpha, \beta, \gamma; x) + Bx^{1-\gamma} F(1-\gamma+\alpha,\ 1-\gamma+\beta,\ 2-\gamma; x),$$

A, B constants.

The solution of (3.9) is therefore

$$\eta(s) = AF(a, b, 1+a+b; is) + B(is)^{-a-b} F(-b, -a, 1-a-b; is).$$

We know that $a, b > 0$ and $\eta(s) \to 1$ as $s \to 0$; consequently $B = 0$ and $A = 1$, yielding

$$\eta(s) = F(a, b, 1+a+b; is).$$

This characteristic function does not correspond to any of the usual distributions. The following proposition identifies the underlying probability law. (N.B. The product of two distributions $\mathscr{L}_1$ and $\mathscr{L}_2$ is denoted $\mathscr{L}_1 \otimes \mathscr{L}_2$. In other words, if X and Y are independent, then $\mathscr{L}(XY) = \mathscr{L}(X) \otimes \mathscr{L}(Y)$.)

Proposition 3.5.1. *Let $0 < a, b < c$ be real numbers. Then the distribution having characteristic function*

$$\eta(s) = Ee^{isZ} = F(a, b, c; is), \quad s \in \mathbf{R}$$

is

$$Z \sim \boldsymbol{\beta}^{(1)}(b, c-b) \otimes \boldsymbol{\Gamma}(a, 1)$$
$$= \boldsymbol{\beta}^{(1)}(a, c-a) \otimes \boldsymbol{\Gamma}(b, 1).$$

Proof. The function $F(a, b, c; is)$ is entirely determined by its derivatives

$$\eta^{(k)}(0) = i^k (a)_k (b)_k / (c)_k, \quad k = 1, 2, \ldots .$$

That all those derivatives exist implies in turn that the moments EZ^k all exist (see Feller, 1971, p. 512 and problem 15, p. 528). Thus

$$EZ^k = (a)_k (b)_k / (c)_k$$
$$= \frac{\Gamma(a+k)\,\Gamma(b+k)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c+k)}, \quad k = 1, 2, \ldots .$$

It is readily verified that a r.v. with distribution $\boldsymbol{\beta}^{(1)}(b, c-b) \otimes \boldsymbol{\Gamma}(a, 1)$ has precisely these moments. By what has been said above the moments uniquely determine this distribution. Finally, the roles of a and b can be reversed in the definition of $F(a, b, c; x)$, see Eq. (3.11). □

Remark 3.5.2. See Chamayou & Letac (1991, Proposition 2.4) for a different proof of Proposition 3.5.1. □

Multiplying the payments by a constant m is the same as multiplying Z by m. This yields the following proposition.

Proposition 3.5.3. *If* $V \sim \boldsymbol{\beta}^{(1)}(a, 1) \otimes \boldsymbol{\beta}^{(1)}(b, 1)$ *and* $C \sim \mathbf{exp}(m)$, *then*

$$Z = \sum_{k \geq 1} V_1 \ldots V_k C_k \sim \boldsymbol{\beta}^{(1)}(b, 1+a) \otimes \boldsymbol{\Gamma}(a, m).$$
$$= \boldsymbol{\beta}^{(1)}(a, 1+b) \otimes \boldsymbol{\Gamma}(b, m)$$

Two final comments will be made. First, this result includes Example 3.4.1 as a limiting case, when we let $b \to \infty$. Since

$$\boldsymbol{\beta}^{(1)}(\alpha, \beta) \to 1 \text{ (degenerate) as } \alpha \to \infty$$

we have on the one hand

$$V \sim \boldsymbol{\beta}^{(1)}(a, 1) \otimes \boldsymbol{\beta}^{(1)}(b, 1) \to \boldsymbol{\beta}^{(1)}(a, 1) \text{ as } b \to \infty$$

and on the other

$$Z \sim \boldsymbol{\beta}^{(1)}(b, 1+a) \otimes \boldsymbol{\Gamma}(a, m) \to \boldsymbol{\Gamma}(a, m) \text{ as } b \to \infty.$$

(For general results concerning the continuity of the distribution of Z with respect to the distributions of C and V, see Theorem 2 of Brandt (1986).)

The second comment concerns the distribution of $V \sim \boldsymbol{\beta}^{(1)}(a, 1) \otimes \boldsymbol{\beta}^{(1)}(b, 1)$. When $a \neq b$

$$P(V\leqslant v)=\int_0^1 P(Y\leqslant v/x)\,ax^{a-1}dx$$
$$=(av^b-bv^a)/(a-b),\ 0<v<1$$

and

$$f_V(v)=(v^{b-1}-v^{a-1})\,ab(a-b)^{-1}1_{(0,1)}(v).$$

In particular, when $a-b=1$

$$f_V(v)=b(b+1)\,v^{b-1}(1-v)\,1_{(0,1)}(v),$$

i.e. $V\sim\beta^{(1)}(b,2)$.

This example will be reinterpreted in paragraph 5.1 (Example 5.1.3).

4. Continuous versions of Z_t and S_t

4.1. This section has to do with discounted or accumulated values when payments are made (and interest compounded) continuously. In the elementary theory of interest, continuous interest is the limit obtained when smaller and smaller payments are made more and more frequently, while interest is also compounded more and more frequently. Paragraph 4.2 applies the same idea to payments and rates of return which are both random. It is shown that, by properly rescaling the distributions of the payments and rates of return, the accumulated values obtained in the limit form a diffusion. Discounted values are also examined. Paragraph 4.3 deals with the moments of these processes, and paragraph 4.4 is concerned with continuous perpetuities (existence theorems and one example of explicit distribution). Finally paragraph 4.5 shows the relationship between continuous perpetuities and the stationary distribution of certain diffusions.

4.2. For each n, define the processes

$$S_n(t)=\sum_{j=0}^{[nt]-1} C_{nj}\,U_{nj+1}\cdots U_{n[nt]} \tag{4.1}$$

$$Z_n(t)=\sum_{j=1}^{[nt]} C_{nj}\,V_{n1}\cdots V_{nj} \tag{4.2}$$

where $[x]$=greatest integer smaller than or equal to x. The subscript "nj" stands for "n,j". The actuarial interpretation of (4.1) is that payment C_{nj} is made at time j/n, and accumulated at rates $R_{nj+1}=U_{nj+1}-1,\ldots,R_{n[nt]}=U_{n[nt]}-1$ up to time t. R_{nj+1} is therefore the rate of return during the period $(j/n,(j+1)/n)$. Expression (4.2) is the corresponding discounted value with payments made at the end of each period rather than at the beginning.

$V_{nj}=1/U_{nj}$ is the discount factor for period $((j-1)/n, j/n)$. The following assumptions are made.

I. For each n, $\{C_{nj}, j\geq 1\}$ and $\{U_{nj}, j\geq 1\}$ are independent i.i.d. sequences.

II. $C_{n1} \stackrel{d}{=} n^{-1}EC_{11}+n^{-1/2}(C_{11}-EC_{11})$, $Var\, C_{11}<\infty$.

III. The distribution of the growth factors $\{U_{nj}\}$ is defined as either

A. $U_{n1} \stackrel{d}{=} 1+n^{-1}E(U_{11}-1)+n^{-1/2}(U_{11}-EU_{11})$, or

B. $\log U_{n1} \stackrel{d}{=} n^{-1}E\log U_{11}+n^{-1/2}(\log U_{11}-E\log U_{11})$.

It is assumed that $P(U_{11}>0)=1$, Var $U_{11}<\infty$ and Var log $U_{11}<\infty$.

Assumptions II and III mean that the distribution of the payments $\{C_{nj}\}$ and growth factors $\{U_{nj}\}$ are obtained from those of the initial variables C_{11} and U_{11}. The payments made in one time unit, e.g. $\Sigma_{j=1}^{n} C_{nj}$, have constant mean and variance. For the growth factors two rescalings are proposed. The first one leaves constant the first two moments of the sum of the arithmetic returns over one period, e.g. $\Sigma_{j=1}^{n} R_{nj}$. The second one does the same for the geometric rates log U_{nj}. In the following propositions, weak convergence is to be understood in the sense of the Skorohod topology on $D[0, T]$ for arbitrary $T>0$. It follows that convergence takes place in $D[0, \infty)$ (e.g. Theorem 23, p. 108, of Pollard, 1984).

Proposition 4.2.1. *Let X, Y be independent standard Brownian motions. $\{S_n, n\geq 1\}$ converge weakly to S satisfying*

$$S_t = \int_0^t \exp\{\tilde{X}_t-\tilde{X}_s\}\, d\tilde{Y}_s \tag{4.3}$$

where

$$\tilde{X}_t = \gamma t+\sigma X_t \tag{4.4}$$

$$\tilde{Y}_t = \mu t+\varrho Y_t \tag{4.5}$$

$$\mu = EC_{11}, \varrho^2 = \text{Var}\, C_{11} \tag{4.6}$$

$$\text{III. A} \qquad \gamma = E(U_{11}-1)-(1/2)\,\text{Var}\, U_{11}, \quad \sigma^2 = \text{Var}\, U_{11} \tag{4.7}$$

$$\text{III. B} \qquad \gamma = E\log U_{11}, \quad \sigma^2 = \text{Var}\log U_{11}. \tag{4.8}$$

S also satisfies the Itô stochastic differential equation (SDE)

$$dS_t = (\alpha S_t+\mu)\, dt+\sigma S_t\, dX_t+\varrho\, dY_t \tag{4.9}$$

where $\alpha=\gamma+\sigma^2/2$.

Proposition 4.2.2. *With the same notation* $\{Z_n, n\geq 1\}$ *converge to* Z *satisfying*

$$Z_t = \int_0^t \exp(-\tilde{X}_s)\, d\tilde{Y}_s.$$

Eqs. (4.4) *to* (4.8) *are unchanged.*

S is a diffusion, while Z is not (the latter is not markovian). The above results extend Proposition 1 of Dufresne (1989).

Proposition 4.2.3. Let

$$B_t = \int_0^t \exp(-\tilde{X}_t + \tilde{X}_s)\, d\tilde{Y}_s.$$

Then $B_t \stackrel{d}{=} Z_t$ *for each fixed* $t\geq 0$.

Proposition 4.2.4. *Let* W *be standard Brownian motion and* Q *the solution of the Itô SDE*

$$dQ_t = (\alpha Q_t + \mu)\, d_t + (\sigma^2 Q^2 + \varrho^2)^{1/2} dW_t. \tag{4.10}$$

Then, given the same initial condition $Q_0 \stackrel{d}{=} S_0 \in L_2$, *the distributions of* Q *and* S *(as random elements of* $C[0, \infty)$*) are identical.*

(The proofs of the above propositions are in the Appendix.)

Other weak convergence results are possible under different assumptions regarding $\{V_{nj}\}$ and $\{C_{nj}\}$. For payment processes $\tilde{X}_n(t) = \Sigma_{j=1}^{[nt]} C_{nj}$ having a limit with bounded variation, the approach used in Dufresne (1989) will work. More generally, theorems concerning the convergence of $\int f_n\, dX_n$, X_n a martingale, might be used; see for example Szyszkowski (1987). For connections with biology see Guess & Gillespie (1977) and Section 8 of Dufresne (1989).

4.3. Let $\alpha_k = k\gamma + k^2\sigma^2/2$, $\beta_k = k\mu$, $\varepsilon_k = k(k-1)\varrho^2/2$. Suppose $S_0 = 0$. Applying Itô's formula, with $f(x) = x^m$, to Eq. (4.9) we get

$$\begin{aligned} S_t^m = &\int_0^t \left[(\mu + \alpha S_u)\, m S_u^{m-1} + \frac{1}{2} m(m-1)\, S_u^{m-2} (\sigma^2 S_u^2 + \varrho^2) \right] du \\ &+ \int_0^t (\ldots)\, dX_u + \int_0^t (\ldots)\, dY_u. \end{aligned} \tag{4.11}$$

After taking expectations on both sides,

$$\frac{d}{dt} ES_t^m = \alpha_m ES_t^m + \beta_m ES_t^{m-1} + \varepsilon_m ES_t^{m-2}. \tag{4.12}$$

The arguments which will be used now are identical to those used in paragraph 2.2; instead of linear difference equations, the moments now satisfy linear differential equations. First, suppose the following conditions are fulfilled:

$$\alpha_i \neq \alpha_j, \quad 0 \leq i < j \leq m. \tag{4.13}$$

Then

$$ES_t^m = \sum_{j=0}^{m} d_{mj} e^{\alpha_j t} \tag{4.14}$$

where d_{mj}, $0 \leq j \leq m$, are constants. A recursive relationship for these constants is found by substituting (4.14) into (4.12). We obtain

$$d_{mj} = \frac{1}{\alpha_j - \alpha_m}(\beta_m d_{m-1,j} + \varepsilon_m d_{m-2,j}), \; 0 \leq j \leq m-2,$$

$$d_{m,m-1} = \frac{1}{\alpha_{m-1} - \alpha_m} \beta_m d_{m-1,m-1}, \; d_{mm} = -\sum_{j=0}^{m-1} d_{mj}.$$

The first two moments of S_t are

$$ES_t = \beta_1(e^{\alpha_1 t} - 1)/\alpha_1,$$

$$ES_t^2 = \beta_1 \beta_2/\alpha_1 \alpha_2 - \varepsilon_2/\alpha_2$$
$$+ [\beta_1 \beta_2/\alpha_1(\alpha_1 - \alpha_2)] e^{\alpha_1 t} + [\varepsilon_2/\alpha_2 - \beta_1 \beta_2/\alpha_1 \alpha_2 + \beta_1 \beta_2/\alpha_1(\alpha_2 - \alpha_1)] e^{\alpha_2 t}.$$

An explicit expression for the constants d_{mj} has only been found for the case $\varrho = 0$:

$$d_{mj} = \mu^m m! / \prod_{\substack{i=0 \\ i \neq j}}^{m} (\alpha_j - \alpha_i). \tag{4.15}$$

Eq. (4.15) holds when conditions (4.13) are satisfied. The proof of (4.15) is in Section 5 of Dufresne (1989).

When condition (4.13) are not satisfied, i.e. when there are $i \neq j$ such that $\alpha_i = \alpha_j$, the expression for ES_t^m contains a term of the form: (constant)$\times t\, e^{\alpha_i t}$. For example, when $\alpha_1 = \alpha_0$, i.e. $\alpha_1 = 0$, $ES_t = \beta_1 t$.

The moments of Z_t follow from Proposition 4.2.3. Let $\delta_k = k\gamma - k^2\sigma^2/2$. If

$$\delta_i \neq \delta_j, \quad 0 \leq i < j \leq m$$

then

$$EZ_t^m = \sum_{j=0}^{m} b_{mj} e^{-\delta_j t},$$

$$b_{mj}=\frac{1}{\delta_m-\delta_j}(\beta_m b_{m-1,j}+\varepsilon_m b_{m-2,j}), \quad 0\leqslant j\leqslant m-2,$$

$$b_{m,m-1}=\frac{1}{\delta_m-\delta_{m-1}}\beta_m b_{m-1,m-1}, b_{mm}=-\sum_{j=0}^{m-1} b_{mj}.$$

If $T\geqslant 0$ is independent of (X, Y) and

$$\bar{A}(\eta)=Ee^{-\eta T}$$

then the mth moment of the life annuity Z_T is

$$EZ_T^m=\sum_{j=0}^{m} b_{mj}\bar{A}(\delta_j).$$

Remark 4.3.1. When $\sigma=0$ (i.e. when rates of return are constant), S_t and Z_t are normal random variables with moments easy to determine. The mean and variance of S_t and Z_t can also be derived when $\sigma=0$ and $\bar{Y}$ is only assumed to have independent increments; see Proposition 2.2 of Harrison (1977). □

4.4. Let us now turn to continuous perpetuities.

Proposition 4.4.1. *Let*

$$Z_t=\int_0^t e^{-\bar{X}_s}d\bar{Y}_s$$

where $\bar{X}_s=\gamma s+\sigma X_s$, $\bar{Y}_s=\mu s+\varrho Y_s$, *X and Y two Brownian motions (not necessarily independent). If* $\gamma>0$ *then* Z_t *converges a.s. as* $t\to\infty$.

Proof.

$$Z_t=\mu\int_0^t e^{-\bar{X}_s}ds+\varrho\sum_{n=0}^{[t]-1}\int_n^{n+1} e^{-\bar{X}_s}dY_s+\varrho\int_{[t]}^t e^{-\bar{X}_s}dY_s.$$

The first term converges almost surely, since there exist $0<\varepsilon<\gamma$ and $T(\omega)$ such that

$$e^{-\bar{X}_s/s}\leqslant e^{\varepsilon-\gamma}, s\geqslant T(\omega).$$

To show that the second term has a finite limit, use the root test. Let $0<\varepsilon<\gamma$ and

$$\mathscr{E}_n=\left\{\left|\int_n^{n+l} e^{-\bar{X}_s}dY_s\right|^{1/n}\leqslant e^{(\varepsilon-\gamma)/2}\right\}.$$

It will be shown that $P(\limsup \mathscr{E}_n)=0$. This implies that only a finite number of the $\{\mathscr{E}_n\}$ occur, that is to say there exists $n(\omega)$ such that

$$\left|\int_n^{n+1} e^{-\tilde{X}_s} dY_s\right|^{1/n} \leq e^{(\varepsilon-\gamma)/2} < 1, \quad n \geq n(\omega).$$

I will use the inequality (Gihman and Skorohod, 1972, p. 11)

$$P\left(\left|\int_0^t f(s)\,dY_s\right| > a\right) \leq P\left(\int_0^t f(s)^2\,ds > b\right) + b/a^2$$

for $a, b>0$. By setting $a=e^{n(\varepsilon-\gamma)/2}$ and $b=e^{-\gamma n}$,

$$P\mathscr{E}_n \leq P\left(\int_n^{n+1} e^{-2\tilde{X}_s} ds > e^{-\gamma n}\right) + e^{-n\varepsilon}.$$

Now

$$\int_n^{n+1} e^{-2\tilde{X}_s} ds \leq e^{-2\gamma n} \exp\left\{-2\sigma \inf_{0\leq s\leq n+1} X_s\right\}$$

which implies

$$\begin{aligned} P\left(\int_n^{n+1} e^{-2\tilde{X}_s} ds > e^{-\gamma n}\right) &\leq P\left(-2\sigma \inf_{0\leq s\leq n+1} X_s > \gamma n\right) \\ &= 2P(X_{n+1} < -\gamma n/2\sigma) \\ &= (2/\pi)^{1/2} \int_{-\infty}^{-\gamma n/2\sigma} e^{-x^2/2} dx. \end{aligned}$$

These equalities result from the reflection principle (e.g. Loève, 1978, p. 261). Since

$$\int_{-\infty}^{-y} e^{-x^2/2} dx < y^{-1} e^{-y^2/2}, y > 0$$

(Feller, 1968, p. 175) we conclude that

$$\sum_{n=1}^{\infty} P\mathscr{E}_n < \infty$$

and so $P(\limsup \mathscr{E}_n)=0$ by the Borel-Cantelli lemma. It remains to show that $\int e^{-\tilde{X}_s} 1_{([t],t)}(s) dY_s$ has a finite limit as $t \to \infty$. The inequality

$$P\left(\sup_{0\leq t\leq T}\left|\int_0^T f(s)\,dY_s\right| > a\right) \leq P\left(\int_0^T f(s)^2 ds > b\right) + b/a^2$$

is a refinement of the previous one (Gihman & Skorohod, 1972, p. 20). It implies

$$P\left(\sup_{n\le t\le n+1}\left|\int_n^t e^{-\tilde{X}_s}dY_s\right|>\varepsilon\right)\le P\left(\int_n^{n+1} e^{-2\tilde{X}_s}ds>e^{-\gamma n}\right)+e^{-\gamma n}/\varepsilon^2.$$

Using the Borel-Cantelli lemma once again, we find that

$$P\left(\limsup_{t\to\infty}\left|\int_{[t]}^t e^{-\tilde{X}_s}dY_s\right|\le\varepsilon\right)=1$$

for every $\varepsilon>0$. □

For processes $\tilde{x}_t=d\tilde{X}_t/dt$ and $\tilde{y}_t=d\tilde{Y}_t/dt$ which are not white noise, Proposition 3.2.2 has the following continuous analogue.

Proposition 4.4.2. *If $(\tilde{x},\tilde{y})$ are stationary and ergodic, and if*

$$E\tilde{x}_t>0, E(\log|\tilde{y}_t|)_+<\infty$$

then

$$Z_t=\int_0^t \exp\left(-\int_0^s \tilde{x}_u\,du\right)\tilde{y}_s\,ds \tag{4.16}$$

converges absolutely a.s. as $t\to\infty$.

Proof. From the hypotheses there exist $a>0$ and T (depending on ω) such that

$$|\tilde{y}_t e^{-\tilde{X}_t}|^{1/t}\le e^{-a},\quad t\ge T$$

and so $\tilde{y}_t e^{-\tilde{X}_t}$ is integrable over $[0,\infty)$. □

Proposition 3.2.3 also admits a continuous counterpart.

Proposition 4.4.3. *Let $W^{(1)}$, $W^{(2)}$ be Brownian motions and*

$$\tilde{x}_t=\gamma t+\sigma_1 W_t^{(1)}$$
$$\tilde{y}_t=\mu t+\sigma_2 W_t^{(2)}$$

If $\gamma>0$ then Z_t in (4.16) converges a.s.

Proof. $\tilde{x}_t\to\infty$ a.s. implies $t^{-1}\int_0^t x_s\,ds\to\infty$ a.s.; the law of the iterated logarithm implies $|\tilde{y}|^{1/t}\to 1$. Thus $[|\tilde{y}_t|\exp(-\int_0^t x_s\,ds)]^{1/t}\to 0$ a.s. □

Finally, let us hark back to the case $\tilde{X}_t=\gamma t+\sigma X_t$, X_t Brownian motion, with constant payments $d\tilde{Y}_t=dt$:

$$Z=\int_0^\infty e^{-\tilde{X}_t}dt.$$

Proposition 4.4.4.

(a) If $\gamma \leqslant 0$ *then* $Z=\infty$ *a.s.*
(b) If $\gamma > 0$ *then* $Z^{-1} \sim \Gamma(2\gamma/\sigma^2, \sigma^2/2)$.

Proof. (a) $Z_t \geqslant \int_0^t e^{-\sigma X_s} ds$

$$\geqslant \lambda\{s \in [0,t]: X_s \leqslant 0\} = M_t$$

where λ is Lebesgue measure. The arc sine law says that $M_t/t \sim \beta^{(1)}(1/2, 1/2)$, which implies ($a \geqslant 0$)

$$P(Z_t \geqslant a) \geqslant \frac{1}{\pi} \int_{1 \wedge a/t}^{1} x^{-1/2}(1-x)^{-1/2} dx$$
$$\to 1 \text{ as } t \to \infty.$$

Thus $Z_t \xrightarrow{P} \infty$ as $t \to \infty$. There is a sequence $\{t_k, k \geqslant 1\}$ such that $t_k \to \infty$ and $Z_{t_k} \xrightarrow{\text{a.s.}} \infty$ as $k \to \infty$. From the fact that Z_t is a.s. non decreasing we conclude that $Z_t \to \infty$ a.s.

(b) Let

$$Z_n(t) = n^{-1} \sum_{j=1}^{[nt]} V_{n1} \dots V_{nj}$$

where $\{V_{nj}, j \geqslant 1\}$ is i.i.d.. The idea of the proof is to choose $\{V_{n1}, n \geqslant 1\}$ in such a way as to (1) know the distribution of $Z_n(\infty)$ for each n, and (2) be able to show that $Z_n \xrightarrow{d} Z$. The only remaining step is proving that $Z_n(\infty) \xrightarrow{d} Z(\infty)$.

Let $V_{nj} \overset{d}{=} A_n/B_n$ with $A_n \sim \Gamma(a_n, 1)$, $B_n \sim \Gamma(b_n, 1)$, A_n and B_n independent, $a_n = 2n/\sigma^2$, $b_n = 2n/\sigma^2 + 2\gamma/\sigma^2$.

Lemma 4.4.5. *Let X be Brownian motion. Then*

$$\left\{ G_n(t) = \prod_{j=1}^{[nt]} V_{nj}, 0 \leqslant t \leqslant T \right\} \xrightarrow{d} \{\exp(-\gamma t - \sigma X_t), 0 \leqslant t \leqslant T\}.$$

Proof of the Lemma. From Eq. (3.8)

$$EG_n(t)^\tau = \{[\Gamma(a_n+\tau)/\Gamma(a_n)][\Gamma(b_n-\tau)/\Gamma(b_n)]\}^{[nt]}.$$

Using the formula (Abramowitz and Stegun, 1965, Eq. 6.1.47, p. 257)

$$\Gamma(x+\tau)/\Gamma(x) = x^\tau[1+\tau(\tau-1)/2x + \mathcal{O}(x^{-2})] \quad \text{as } x \to \infty$$

we get

$$EG_n(t)^\tau = (a_n/b_n)^{\tau[nt]}[1+\tau(\tau-1)/2a_n]^{[nt]}[1+\tau(\tau+1)/2b_n]^{[nt]}[1+\mathcal{O}(n^{-2})]^{[nt]}$$
$$\rightarrow \exp[-\gamma t\tau+\sigma^2 t\tau^2/2].$$

Thus $\tilde{X}_n(t)=-\log G_n(t)\xrightarrow{d}\mathbf{N}(\gamma t, \sigma^2 t)$ for each $t>0$. Similarly $\tilde{X}_n(t)-\tilde{X}_n(s)$ converges in distribution to $\mathbf{N}(\gamma(t-s), \sigma^2(t-s))$, for all $0\leq s<t$. To prove that the whole process $\tilde{X}_n=\{\tilde{X}_n(t),\ 0\leq t\leq T\}$ converges weakly to $\tilde{X}=\{\tilde{X}_t=\gamma t+\sigma X_t,\ 0\leq t\leq T\}$, one possibility is to use Theorem 19, p. 104, of Pollard (1984). It is sufficient to show that for every $\delta>0$ there exist $\alpha>0$, $\beta>0$, $n_0\in\mathbf{N}$ such that

$$P\{\tilde{X}_n(t)-\tilde{X}_n(s)|\leq\delta\}\geq\beta$$

whenever $|t-s|<\alpha$ and $n\geq n_0$.

From Markov's inequality

$$P\{|\tilde{X}_n(t)-\tilde{X}_n(s)|\leq\delta\}\geq 1-E[X_n(t)-X_n(s)]^2/\delta^2$$
$$\rightarrow 1-[\sigma^2|t-s|+\gamma^2(t-s)^2]/\delta^2.$$

This completes the proof of the lemma. □

From the foregoing lemma and Lemma 3 of Dufresne (1989), we conclude that the processes $Z_n=\{Z_n(t), t\geq 0\}$ converge weakly (as random elements of $D[0,\infty)$) to $Z=\{Z(t)=\int_0^t e^{-\tilde{X}_s}ds,\ t\geq 0\}$. On the other hand, Example 3.4.3 tells us that

$$nZ_n(\infty)=\lim_{t\to\infty} nZ_n(t)\sim\boldsymbol{\beta}^{(2)}(a_n, b_n-a_n)$$

or, equivalently,

$$Z_n(\infty)\stackrel{d}{=}C_n/D,\ (C_n, D)\text{ independent,} \tag{4.17}$$
$$C_n\sim\boldsymbol{\Gamma}(a_n, n^{-1}),\ D\sim\boldsymbol{\Gamma}(2\gamma/\sigma^2,\ 1).$$

Once it is proved that $Z_n(\infty)\xrightarrow{d}Z(\infty)$ (see below), we will be able to conclude that the distribution of $Z(\infty)=\int_0^\infty e^{-\tilde{X}_s}ds$ is the limit as $n\to\infty$ of (4.17). Clearly C_n converges to the constant $2/\sigma^2$; hence

$$D/C_n\xrightarrow{d}\boldsymbol{\Gamma}(2\gamma/\sigma^2, \sigma^2/2).$$

To prove that $Z_n(\infty)\xrightarrow{d}Z(\infty)$, use the Extended Skorohod Representation Theorem (e.g. Pollard, 1984, p. 71). According to that theorem, $Z_n\xrightarrow{d}Z$ implies that there exist $\tilde{Z}_n\stackrel{d}{=}Z_n$, $\tilde{Z}\stackrel{d}{=}Z$, such that $\tilde{Z}_n\xrightarrow{\text{a.s.}}\tilde{Z}$. Here the metric on $D[0,\infty)$ is the uniform metric on bounded intervals. This is possible because the limit process Z is a.s. continuous. Fix ω. There almost surely exists $T = \mathrm{T}(\varepsilon, \omega)$ such that

$$|\tilde{Z}_n(\infty) - \tilde{Z}(\infty)| \leqslant \sup_{0 \leqslant t \leqslant T} |\tilde{Z}_n(t) - \tilde{Z}(t)| + 2\varepsilon$$

since $\tilde{Z}_n(t)$ and $\tilde{Z}(t)$ both converge as $t \to \infty$. Letting $n \to \infty$ on the r.h.s. yields $\tilde{Z}_n(\infty) \xrightarrow{\text{a.s.}} \tilde{Z}(\infty) \;\Rightarrow\; Z_n(\infty) \xrightarrow{d} Z(\infty)$. □

4.5. The duality between accumulating and discounting (Proposition 4.2.3) will now be employed to translate Propositions 4.4.1 and 4.4.4 into results concerning the stationary distribution of certain diffusions.

Proposition 4.5.1.

(*a*) *A process satisfying the Itô SDE*

$$dB_t = (aB_t + \mu)\,dt + \sigma B_t\,dX_t + \varrho\,dY_t, \tag{4.18}$$

with X, Y independent Brownian motions, has a stationary distribution if $a < \sigma^2/2$.

(*b*) *If* $\varrho = 0$, *a stationary distribution exists if and ony if* $a < \sigma^2/2$, *in which case it is* $\Gamma(1 - (2a/\sigma^2), \sigma^2/2\mu)^{-1}$.

(*c*) *Part* (*a*) *also holds for B satisfying* $dB_t = (aB_t + \mu)\,dt + (\sigma^2 B_t^2 + \varrho^2)^{1/2} dW_t$, *W Brownian motion.*

Proof. (*a*) Eq. (2.8) can be extended to

$$Z_t + HV_1 \ldots V_t \stackrel{d}{=} KV_1 \ldots V_t + B_t,\; H \stackrel{d}{=} K,$$

where H and K are independent of $\{(V_k, C_k), 0 \leqslant k \leqslant t\}$, Z_t is given by (2.7) and B_t is the r.h.s. of (2.8). By taking limits as in paragraph 4.2 we obtain

$$\int_0^t e^{-\tilde{X}_s} d\tilde{Y}_s + He^{-\tilde{X}_t} \stackrel{d}{=} Ke^{-\tilde{X}_t} + e^{-\tilde{X}_t} \int_0^t e^{\tilde{X}_s} d\tilde{Y}_s \tag{4.19}$$

where $\tilde{X}$, $\tilde{Y}$ are as in Proposition 4.2.1. Here H and K are independent of $\{(\tilde{X}_s, \tilde{Y}_s), 0 \leqslant s \leqslant t\}$. Let $\gamma > 0$. With the choice

$$H = \int_t^\infty e^{-(\tilde{X}_s - \tilde{X}_t)} d\tilde{Y}_s$$

$$\stackrel{d}{=} \int_t^\infty e^{-\tilde{X}_s} d\tilde{Y}_s \tag{4.20}$$

the l.h.s. of (4.19) is equal to (4.20). Therefore the r.h.s. of (4.19), which is the solution of *SDE* (4.18) when $\gamma = -a + \sigma^2/2$ and $B_0 = K$, has the same distribution as (4.20) as soon as K does. Moreover this distribution is always obtained as $t \to \infty$, for any $B_0 \in L_2$.

(*b*) Apply Proposition 4.4.4 with $\gamma = -a + \sigma^2/2$.

(*c*) Apply Proposition 4.2.4. □

These results can also be proved using the analytical theory of Markov processes, see for example Chapter 4 of Mandl (1968) or Chapter 15 of Karlin and Taylor (1981). The alternative proof given here is more probabilistic than analytical.

A problem related to the above proposition is whether the process

$$B_t = e^{-\gamma t}\int_0^t e^{\gamma s}\,dA(s), \quad \gamma > 0$$

converges in distribution, when A has independent and homogeneous increments. This has been studied by Harrison (1977) and Wolfe (1982).

A final observation will be made regarding the intuitive interpretation of some stochastic differential equations. It is sometimes said that in Itô *SDE*s of the form

$$dB_t = (aB_t + 1)\,dt + \sigma B_t\,dW_t \tag{4.21}$$

the instantaneous growth rate has "mean" a and "variance" σ^2. When $a>0$ this should imply $B_t \to \infty$ a.s., which is not always the case: B_t has a stationary (or "equilibrium") distribution even when $0 \leqslant a < \sigma^2/2$. To restore comparability with ordinary differential equations it is useful (at least intuitively) to rewrite (4.21) as a Stratonovich equation (Stratonovich, 1966)

$$(S)\,dB_t = \left[(a-\sigma^2/2)\,B_t + 1\right]dt + \sigma B_t\,dW_t.$$

Another way of putting this is: when introducing white noise growth rates into an ordinary differential equation, one should first interpret the *SDE* obtained in Stratonovich sense, and then transform it into an Itô *SDE*. For more on this topic see Arnold (1974), pp. 172–175.

5. Applications to risk theory

5.1. Suppose i.i.d. claims $\{C_k, k \geqslant 1\}$ occur at times $\{T_k, k \leqslant 1\}$, the set $\{T_k\}$ forming a Poisson point process on $\mathbf{R}^+$. The r.v.'s $\{C_k\}$ and $\{T_k\}$ are assumed independent. If the discounted value at time 0 of a claim of amount c occuring at time t is $g(t,c)$, then

$$D_t = \sum_{0 < T_k \leqslant t} g(T_k, C_k) \tag{5.1}$$

represents the discounted value of claims up to time t. The discounted value of all future claims is

$$D = D_\infty = \sum_{k \geqslant 1} g(T_k, C_k) \tag{5.2}$$

whenever the sum on the right converges.

I first review some of the existing literature concerning the random variables D and D_t. Then two examples of perpetuities from Section 3 are reinterpreted in the present context. A good part of Section 3 also applies to the study of r.v.'s such as (5.2).

A sufficient condition for the convergence of (5.2) is easily found. Let λ be the parameter of the Poisson point process. Then

$$\begin{aligned} E\sum_{k\geq 1}|g(T_k,C_k)| &= \sum_{k\geq 1}\int\int_0^\infty |g(t,x)|\frac{\lambda^k t^{k-1}}{(k-1)!}e^{-\lambda t}dt\,dF_C(x) \\ &= \lambda\int\int_0^\infty |g(t,x)|\,dt\,dF_C(x) \\ &= \lambda\int_0^\infty E|g(t,C)|dt. \end{aligned}$$

Consequently, D_t will converge absolutely to a finite D (a.s. and in L_1) as soon as

$$\int_0^\infty E|g(t,C)|dt<\infty. \tag{5.3}$$

This result was obtained by Takacs (1954, 1955, 1956) in a series of papers on shot noise. Suppose signals $\{C_k\}$ occur at times $\{U_k\}$, these moments forming a Poisson point process on the whole line. A counter records those signals. The amplitude recorded changes over time according to a function $g(t,x)$, i.e. a signal with original amplitude x leaves an impression $g(t,x)$ t time units after it's recorded. The total impression recorded by the counter at time u, with respect to signals occuring in $[u-t,u)$, is therefore

$$\zeta(u,t)=\sum_{u-t\leq U_k<u} g(u-U_k,C_k).$$

Takacs showed that

$$\log E\,exp[i\sigma\zeta(u,t)]=\lambda\int_0^t[\varphi(v,\sigma)-1]\,dv \tag{5.4}$$

where

$$\varphi(v,\sigma)=E\exp[i\sigma g(v,C)].$$

He also showed that condition (5.3) is sufficient for the existence of the limit of $\zeta(u,t)$ as $t\to\infty$.

The relationship between $\zeta(u,t)$ and D_t is easy to see:

$$\{\zeta(u,t),t\geq 0\}\stackrel{d}{=}\{D_t,t\geq 0\}.$$

This may be viewed as another instance of duality between accumulating and discounting: $\zeta(u, t)$ represents the value at time u of amounts C_k accumulated from moments U_k to u, if the accumulating factor is $g(u-U_k, C_k)$. Consequently

$$\log E \exp(i\sigma D_t) = \lambda \int_0^t [\varphi(v, \sigma)-1]\, dv. \tag{5.5}$$

Takacs' results are classical in the theory of stochastic processes: for example, (5.4) is derived in Karlin & Taylor (1975, pp. 128–131) and Ross (1983, pp. 212–213), in the case of exponentially decaying signals

$$g(u, x) = xe^{-\delta u}. \tag{5.6}$$

Formula (5.5) was also obtained by G. C. Taylor (1979). Gerber (1979, p. 136) mentions special case (5.6) when $t=\infty$:

$$\log E \exp(i\sigma D) = \lambda \int_0^\infty [E \exp(i\sigma e^{-\delta v} C)-1]\, dv. \tag{5.7}$$

Here condition (5.3) for the a.s. convergence of D amounts to $E|C|<\infty$. From the discussion in Section 3 and Eq. (5.10) below, it is seen that this condition can be weakened.

Vervaat (1979) considers

$$D = \sum_{k=1}^{\infty} C_k f(T_k) \tag{5.8}$$

(i.e. $g(t, x)=xf(t)$) with $f\geqslant 0$ non-increasing and right-continuous. He proves the following necessary and sufficient condition for the absolute convergence of (5.8). (Without loss of generality C is assumed non-negative.)

Proposition 5.1.1. (Vervaat). *Let $\bar{f}$ be the right-continuous inverse of f with $\int_1^\infty f(x)\, dx<\infty$. Then* (5.8) *converges if and only if*

$$E \int_0^1 \bar{f}(x/C)\, dx < \infty.$$

Two more recent papers have dealt with (5.1) when $g(t, x)=xf(t)$. Boogaert et al. (1988) rederive (5.5) and condition (5.3) for the case $f\geqslant 0$ continuous and non-increasing. They get (5.5) from

$$D_t = \sum_{k=1}^{N_t} C_k f(T_k) \overset{d}{=} \sum_{k=1}^{N_t} C_k f(Y_k^{(t)}) \tag{5.9}$$

where $\{Y_n^{(t)}, n\geqslant 1\}$ is an i.i.d. sequence of U[0, t] variables, independent of $\{C_k\}$. This is the technique Takacs used in the appendix of his 1954 paper.

Relationship (5.9) expresses the "order statistic" property of the Poisson point process. Willmot (1989) studies the discounted claims process when the discount factor is a positive function f, and when the claim number process (not necessarily Poisson) possesses the order statistic property.

Finally, Waters (1983) considers a discrete-time version of the Poisson model studied by G. C. Taylor (1979).

Example 5.1.2. Suppose $g(t,x)=xe^{-\delta t}$, $\{T_k\}$ is Poisson with parameter λ and $\delta>0$. Then

$$D=\sum_{k\geqslant 1} C_k e^{-\delta T_k}$$

$$=\sum_{k\geqslant 1} V_1\ldots V_k C_k, \qquad (5.10)$$

$$V_j=\exp[-\delta(T_j-T_{j-1})]\sim\boldsymbol{\beta}^{(1)}(\lambda/\delta,1).$$

D is identical to the variable Z of Example 3.4.1. Eqs (3.3) and (5.7) are equivalent (let $u=se^{-\delta t}$, $a=\lambda/\delta$), and $C\sim\mathbf{exp}(1)$ implies $D\sim\boldsymbol{\Gamma}(\lambda/\delta,1)$.

Takacs also worked out two other cases: (1) $C\sim\boldsymbol{\Gamma}(n,1)$, $n\in\mathbf{N}^+$ (1954 paper), (2) $C\equiv 1$ (1955 paper). In the first case

$$\log E\,e^{i\sigma D}=(\lambda/\delta)\int_0^\sigma [(1-iu)^{-n}-1]/u\,du$$

$$=(i\lambda/\delta)\int_0^\sigma \sum_{j=1}^{n}(1-iu)^{-j}du$$

$$=\log(1-i\sigma)^{-\lambda/\delta}+\sum_{k=1}^{n-1}(\lambda/k\delta)[(1-i\sigma)^{-k}-1].$$

Here D has the distribution of a sum of n independent variables, the first one having a gamma distribution and each of the other $n-1$ having a compound Poisson distribution.

Oddly enough the degenerate case $C\equiv 1$ is a lot more difficult. Other references on this subject are Vervaat (1972, p. 90) and Lassner (1974). □

Example 5.1.3. The perpetuity in paragraph 3.5 can also be interpreted as the discounted value of future claims occuring at renewal times $\{T_k\}$. If $T_0=0$ and the distribution of T_k-T_{k-1} is the sum of two independent exponential distributions with parameters $1/\lambda_1$ and $1/\lambda_2$, then

$$\exp-\delta(T_k-T_{k-1})\overset{d}{=}XY$$

where X and Y are independent and have distributions $\boldsymbol{\beta}^{(1)}(\lambda_1/\delta,1)$ and $\boldsymbol{\beta}^{(1)}(\lambda_2/\delta,1)$, respectively. Eq. (5.10) is obtained, with $V_k\overset{d}{=}XY$. The characteristic function of D therefore satisfies (3.9). Finally

$$C\sim\mathbf{exp}(1)\Rightarrow D\sim\boldsymbol{\Gamma}(\lambda_1/\delta,1)\otimes\boldsymbol{\beta}^{(1)}(\lambda_2/\delta,1+\lambda_1/\delta).$$

Observe that in this example the claim number process does not possess the order statistic property: the only renewal process having this property is the Poisson process (Liberman, 1985). □

5.2. This paragraph is about approximating risk processes by diffusion processes. For the classical no-interest no-inflation risk process

$$S_t = \sum_{i=1}^{N_t} X_i,$$

Iglehart (1969) and Grandell (1977, 1978) have suggested a Brownian motion with drift:

$$dZ_t = m dt - \varrho dY_t,$$

Y standard Brownian motion. If premiums are received at constant rate $c=\mu+m$, the risk reserve process then satisfies

$$dU_t = \mu dt + \varrho dY_t. \tag{5.11}$$

Emmanuel et al. (1975), Harrison (1977), Ruohonen (1980) and Braun (1986) have modified this process to include a constant force of interest γ by letting

$$dU_t = (\mu + \gamma U_t)\, dt + \varrho dY_t.$$

Garrido (1988) introduced varying (but deterministic) interest and inflation into model (5.11).

In light of Section 4, the model

$$U_t = u e^{\tilde{X}_t - \tilde{X}_0} + \int_0^t e^{\tilde{X}_t - \tilde{X}_s} d(\mu s + \varrho Y_s) \tag{5.12}$$

can be suggested as a generalization of (5.11) including stochastic returns on assets. Here $\exp(\tilde{X}_t - \tilde{X}_s)$ is the accumulation factor from time s to time t. Assuming $\tilde{X}$ and Y to be independent (this may not always be realistic) it is possible to calculate the mean and variance of U_t.

The study of (5.12) becomes easier when $\tilde{X} = \gamma t + \sigma X_t$, X standard Brownian motion independent of Y. Then

$$dU_t = (\mu + \alpha U_t)\, dt + \sigma U_t dX_t + \varrho dY_t$$

where $\alpha = \gamma + \sigma^2/2$. The moments of U_t are derived from differential equations (4.12). U is a diffusion with infinitesimal mean and variance

$$m(x) = \alpha x + \mu, \quad v(x) = \sigma^2 x^2 + \varrho^2.$$

Therefore U has the same distribution (in $C[0, \infty)$) as the solution of

$$dR_t = (\mu+\alpha R_t)\,dt+(\sigma^2 R_t^2+\varrho^2)^{1/2}dW_t,$$

$$R_0 = U_0 \text{ a.s.}$$

Ruin probabilities can be obtained from the classical theory of barrier crossing for diffusion processes (Darling & Siegert, 1953; Ruohonen, 1980). The discounted surplus process becomes

$$D_t = U_t e^{-\tilde{X}_t}$$
$$= u+\int_0^t e^{-\tilde{X}_s}d\tilde{Y}_s.$$

It is not a diffusion, but from Proposition 4.2.3 $D_t \overset{d}{=} u+B_t$ (t fixed) where

$$dB_t = (\mu-\delta B_t)\,dt+\sigma B_t\,dX_t+\varrho\,dY_t,$$

$$B_0 = 0, \delta = \gamma-\sigma^2/2.$$

Hence all the moments of D_t can also be obtained from (4.12). The limit of D_t is a.s. finite as soon as $\gamma>0$.

6. Applications to pension funding

6.1. Contributions to the deterministic theory of pension funding include Bowers et al. (1976, 1979, 1982), Dufresne (1988*a*), J. R. Taylor (1967), Treuil (1981), Trowbridge (1952, 1963) and Winklevoss (1977). One way of improving the deterministic model is to let rates of return on assets as well as populations be random. Define

AL	= actuarial liability (mathematical reserve)	F	= value of fund
B	= benefit payments	NC	= normal cost (net premium)
C	= total contribution	R	= rate of return.

Assuming that benefit payments and contributions are paid at the beginning of the year,

$$F_t = (1+R_t)(F_{t-1}+C_{t-1}-B_{t-1}). \tag{6.1}$$

Suppose that, once the actuarial basis required to calculate AL and NC is chosen, the total contribution is defined as

$$C_t = NC_t+k(AL_t-F_t)$$

for a certain constant $k\leqslant 1$ (in practice k might be set equal to

$[i/(1+i)]/[1-(1+i)^{-n}]$, i the valuation rate of interest and $n=10$ or 20). Eq. (6.1) then reduces to

$$F_t = V_t(F_{t-1}+H_{t-1}) \tag{6.2}$$

where $V_s=(1-k)(1+R_s)$ and $H_s=(kAL_s+NC_s-B_s)/(1-k)$. Also define

$$F(t,X) = X\prod_{j=1}^{t} V_j+\sum_{i=1}^{t} H_{t-i} \prod_{j=t-i+1}^{t} V_j.$$

Proposition 6.1.1. (Brandt, 1986). *If* $(V_i, H_i; i\in \mathbf{Z})$ *is stationary and ergodic and if*

$$-\infty \leq E\log|V| < 0, E(\log|H|)_+ < \infty$$

then (6.2) *has the unique stationary solution*

$$F_t = \sum_{i=1}^{\infty} H_{t-i} \prod_{j=t-i+1}^{t} V_j, t\in \mathbf{Z}.$$

Furthermore

$$\lim_{t\to\infty} |F_t-F(t,X)| = 0 \quad \text{a.s.}$$

for any r.v. X.

The proof of Proposition 6.1.1 is very similar to that of Proposition 3.2.2: the reader is referred to the original paper for details (see also Proposition 6.2.1).

The above result says that one of the conditions sufficient for F_t to have a limit (in distribution) is that k, the proportion of the unfunded liability constituting the "feedback" into the system, is large enough to imply

$$\log(1-k)+E\log(1+R) < 0.$$

The proposition also applies to the aggregate funding method, see Dufresne (1986). In the case of i.i.d. $\{R_k\}$ and static population it is possible to show that some values of k are better than others. This problem is described in detail in the continuous-time setting of paragraph 6.3. The discrete-time formulation has appeared in Dufresne (1988*b*).

6.2. A continuous-time counterpart of Eq. (6.1) is

$$F_t' = \gamma_t F_t+C_t-B_t \quad \text{a.s.}$$

where γ_t is the instantaneous rate of return and C and B are paid continuously. If C_t is given by

$$C_t = NC_t + k(AL_t - F_t), \quad k \in \mathbf{R} \tag{6.3}$$

then

$$F_t' = -A_t F_t + H_t \tag{6.4}$$

with $A_t = k - \gamma_t$ and $H_t = kAL_t + NC_t - B_t$. The following proposition is the continuous-time replica of Proposition 6.1.1. Its proof parallels that of Theorem 1 of Brandt (1986).

Proposition 6.2.1. *Suppose* $\{A_t, H_t\}$ *are stationary and ergodic and that*

$0 < EA_t \leq \infty$, $E(\log|H_t|)_+ < \infty$.

Then

$$F_t = \int_{-\infty}^{t} \exp\left(-\int_s^t A_u\,du\right) H_s\,ds \tag{6.5}$$

is the unique stationary solution of (6.4). *The r.h.s. of* (6.5) *converges absolutely almost surely. Furthermore, if*

$$F_t(X) = \int_0^t \exp\left(-\int_s^t A_u\,du\right) H_s\,ds + X\exp - \int_0^t A_u\,du \tag{6.6}$$

then

$$\lim_{t\to\infty} |F_t(X) - F_t| = 0 \quad \text{a.s.} \tag{6.7}$$

for any r.v. X.

Proof. For each realization of (A, H) there exist $\varepsilon > 0, s_0 > 0$ such that

$$\left|\exp\left(-\int_s^t A_u\,du\right) H_s\right| \leq e^{-\varepsilon(t-s)}$$

for $s < s_0$. The r.h.s. of (6.5) thus converges absolutely. To prove (6.7), observe that from (6.5) and (6.6)

$$\begin{aligned} |F_t(X) - F_t| &= \exp\left(-\int_0^t A_u\,du\right) |F_0 - X| \\ &\to 0 \text{ as } t \to \infty \quad \text{a.s.} \end{aligned}$$

That the r.h.s. of (6.5) satisfies (6.4) is clear. To prove that it is stationary apply the shift operator $(x_v) \to (x_{v+h})$ to $\{A_u\}$ and $\{H_s\}$ on the r.h.s. of (6.5) to obtain

$$F_t \stackrel{d}{=} \int_{-\infty}^{t} \exp\left(-\int_s^t A_{u+h}\,du\right) H_{s+h}\,ds$$
$$= \int_{-\infty}^{t+h} \exp\left(-\int_s^{t+h} A_u\,du\right) H_s\,ds$$
$$= F_{t+h}.$$

The same argument shows that $(F_{t_1}, \ldots, F_{t_k}) \stackrel{d}{=} (F_{t_1+h}, \ldots, F_{t_k+h})$ for any $\{t_1, \ldots, t_k\}$.

Finally, let G be another stationary solution of (6.4) and let $D=F-G$. Then

$$D_t' = -A_t D_t$$

which implies

$$D_t = \exp\left(-\int_s^t A_u\,du\right)(F_s - G_s).$$

Hence $D_t \xrightarrow{p} 0$ as $s \to -\infty$, since $\exp(-\int_s^t A_u\,du) \to 0$ as $s \to -\infty$ a.s. and F and G are stationary. Thus $D_t=0$ a.s. □

The requirement $EA_t>0$ says that the fraction k of the unfunded liability $(AL_t - F_t)$ included in C_t has to be greater than the average rate of return on assets.

Turn now to rates of return which form a white noise $\gamma+\sigma dW_t/dt$. Suppose, furthermore, that the population is static and that the actuarial valuation basis is unchanged over time. Then NC_t, AL_t and B_t are constants, and Eq. (6.4) becomes

$$dF_t = [(\gamma+\sigma^2/2)F_t + C_t - B]\,dt + \sigma F_t\,dW_t \tag{6.8}$$
$$= (-AF_t + H)\,dt + \sigma F_t\,dW_t \tag{6.9}$$

where $A=k-\gamma-\sigma^2/2$, $H=kAL+NC-B$ and W is standard Brownian motion. Observe that, in view of Section 4, the coefficient of F_t in (6.8) is not the "mean" rate return γ, but rather $\alpha=\gamma+\sigma^2/2$. As in paragraph 4.3, all the moments of F_t (and therefore those of C_t) can be found by applying Itô's formula to (6.9):

$$dEF_t^m/dt = c_m EF_t^m + mHEF_t^{m-1}$$

where $c_m = m(\gamma-k)+m^2\sigma^2/2$. In particular, if $F_0=0$ and $c_i \neq c_j$, $1 \le i < j \le m$, then

$$EF_t^m = H^m \sum_{j=0}^{m} d_{mj} e^{c_j t}$$

where

$$d_{mj} = m! \Big/ \prod_{\substack{i=0 \\ i \neq j}}^{m} (c_j - c_i).$$

From Proposition 4.5.1

$$F_t^{-1} \xrightarrow{d} F_S^{-1} \sim \Gamma(a = 2(k-\gamma)/\sigma^2, b = \sigma^2/2H) \tag{6.10}$$

as $t \to \infty$ iff $k > \gamma$.

This limit distribution is also related to the following economic problem. If the population size is geometric Brownian motion, then under a Cobb–Douglas production function $f(r) = r^\alpha$, $0 < \alpha < 1$ (and other assumptions) the capital-labour ratio $R(t)$ satisfies the Itô *SDE*

$$dR_t = (C_1 R_t^\alpha - C_2 R_t)\, dt - \sigma R_t\, dW_t$$

($C_1, C_2 > 0$). Bourguignon (1974) and Merton (1975) found the limiting distribution of R_t as $t \to \infty$. Observe that if $X = R^{1-\alpha}$ then

$$dX_t = (-C_3 X_t + C_4)\, dt - C_5 X dW$$

($C_3, C_4, C_5 > 0$). The limit distribution of R_t can therefore be obtained from Proposition 4.5.1.

In the actuarial literature, O'Brien (1986, 1987) has studied the evolution of the funded ratio (value of fund over present value of future benefits) when rates of return and population growth rates are white noise processes.

6.3. Let us take a closer look at Eq. (6.9). In the totally static situation ($\sigma^2 = 0$)

$$F_t \to H/A = (kAL + NC - B)/(k - \gamma), \quad k > \gamma. \tag{6.11}$$

It is known from pension mathematics (e.g. Eq. (42) of Bowers et al., 1976) that if γ_v is the valuation rate of interest used to compute AL and NC, then

$$0 = \gamma_v AL + NC - B$$

(this is known as the "equation of equilibrium"). Applying this to (6.11) we find that $F_t \to AL(k - \gamma_v)/(k - \gamma)$. Hence F_t converges to AL iff $\gamma_v = \gamma$, i.e. iff the valuation rate of interest is equal to the rate earned by the fund's assets.

In the stochastic interest case ($\sigma^2 > 0$) there is a.s. no pointwise limit for F_t, but we can look at the mean and mode of the stationary distribution F_S (see (6.10)). From

$$EF_S^m = b^{-m}\Gamma(a-m)/\Gamma(a), \quad m<a \tag{6.12}$$

we get (if $k>\gamma+\sigma^2/2$)

$$EF_S = 1/(a-1)\,b = AL(k-\gamma_v)/(k-\gamma-\sigma^2/2).$$

Thus $EF_S=AL$ iff $\gamma_v=\gamma+\sigma^2/2=\alpha_1$. This is consistent with the fact that funds invested at rates of return $\gamma+\sigma dW_t/dt$ have expected values which grow at rate α_1 (let $m=1$ in (4.12)).

The mode of the stationary distribution is found from the stationary density

$$f(y) = (\text{constant})\cdot y^{-(a+1)}\,e^{-1/by}\,1_{(0,\infty)}(y)$$

and is

$$\begin{aligned}\text{mode} = M &= 1/(a+1)\,b\\ &= AL(k-\gamma_v)/(k-\gamma+\sigma^2/2)\\ &< EF_S.\end{aligned}$$

Hence $M=AL$ iff $\gamma_v=\gamma-\sigma^2/2=\delta_1$.

An obvious question to ask in relation to the present pension model is whether some values of k are better than others. One way of answering this question is to let $\gamma_v=\gamma+\sigma^2/2$ (which implies $EF_S=AL$) and calculate the variance of F and C for different values of k. From (6.12)

$$\begin{aligned}\operatorname{Var} F_S &= 1/[(a-1)^2(a-2)b^2]\\ &= \sigma^2 AL^2/2(k-\gamma-\sigma^2), \quad k>\gamma+\sigma^2.\end{aligned}$$

From (6.3) the variance of the stationary contribution rate (C_S) is therefore

$$\operatorname{Var} C_S = \sigma^2 AL^2k^2/2(k-\gamma-\sigma^2).$$

$\operatorname{Var} F_S$ is a decreasing function of k; this agrees with intuition. The situation is less obvious for $\operatorname{Var} C_S$.

Proposition 6.3.1. *Let $\gamma_v=\gamma+\sigma^2/2$ and $k>\gamma+\sigma^2$. Then $EF_t\to AL$, $EC_t\to NC$ and $\operatorname{Var} F_S$ is a decreasing function of k.*
(a) *When $\gamma+\sigma^2\neq 0$, let*

$$k^* = \begin{cases} 0, & \text{if } \gamma+\sigma^2<0\\ 2(\gamma+\sigma^2), & \text{if } \gamma+\sigma^2>0.\end{cases}$$

Then (1) *for $\gamma+\sigma^2<k<k^*$, $\operatorname{Var} C_S$ is a decreasing function of k;*
(2) *for $k>k^*$, $\operatorname{Var} C_S$ is an increasing function of k.*
(b) *When $\gamma+\sigma^2=0$, $\operatorname{Var} C_S$ is an increasing function of k for all $k>0$.*

Proof. Part (a) follows from

$$\partial \operatorname{Var} C_S/\partial k = k[k-2(\gamma+\sigma^2)]\,\sigma^2 AL^2/2(k-\gamma-\sigma^2)^2. \qquad \square$$

The meaning of this proposition is twofold. First, values of k smaller than k^* are suboptimal: both $\operatorname{Var} C_S$ and $\operatorname{Var} F_S$ can be decreased by choosing $k=k^*$ instead. Second, for $k \geqslant k^*$ there is a trade-off between $\operatorname{Var} F_S$ and $\operatorname{Var} C_S$: increasing k will decrease $\operatorname{Var} F_S$ but increase $\operatorname{Var} C_S$. No single value of k in $[k^*, \infty)$ is better than the others. Observe that, when $\gamma+\sigma^2>0, k^*=2(\gamma+\sigma^2)>2\gamma_v$, which means that, in order to avoid inadmissibly high $\operatorname{Var} C$ and $\operatorname{Var} F$, contributions have to include more than twice interest (at the valuation rate) on the unfunded liability.

Proposition 6.3.1 deals with the stationary distribution obtained when $t \to \infty$. For finite values of t there may or may not exist a $k^*(t)$ with the same properties as the above k^*. See Dufresne (1988 *b*) for details on this subject.

Acknowledgements

Support from the National Sciences and Engineering Research Council of Canada is gratefully acknowledged.

Appendix

Proofs of the propositions in Paragraph 4.2

Proof of Proposition 4.2.1. Refer to Gihman & Shorohod (1979)—"GS" in the sequel, pp. 184–190 and 208. I will show that

$$\left\{ \tilde{S}_n(t) = \sum_{j=1}^{[nt]} C_{nj}\, U_{nj+1} \cdots U_{n[nt]},\ t \geqslant 0 \right\}$$

converge weakly to S. Since $\tilde{S}_n(t)-S_n(t)=C_{n[nt]}-C_{n0}\, U_{n1} \cdots U_{n[nt]}$, it follows that $\sup_{0 \leqslant t \leqslant T} |\tilde{S}_n(t)-S_n(t)|$ converges in probability to 0 as $n \to \infty$, which implies that S_n also converges weakly to S; for justification see Theorem 4.1 of Billingsley (1968), or Problem 10, p. 87, of Pollard (1984).

Defining $\tilde{S}_{nk}=\tilde{S}_n(k/n)$ we obtain

$$\Delta \tilde{S}_{nk} = \tilde{S}_{nk+1} - \tilde{S}_{nk} = (\mu+\alpha_n \tilde{S}_{nk})\, n^{-1} + \sigma_n \tilde{S}_{nk}\, \Delta X_{nk} + \varrho \Delta Y_{nk}$$

where

$$\mu = EC_{11},\ \varrho^2 = \operatorname{Var} C_{11},\ \alpha_n = nE(U_{n1}-1),\ \sigma_n^2 = n \operatorname{Var} U_{n1},$$

$$X_{nk} = \sigma_n^{-1} \sum_{j=1}^{k} (U_{nj}-EU_{nj}),\ Y_{nk} = \varrho^{-1} \sum_{j=1}^{k} (C_{nj}-EC_{nj}).$$

Furthermore, define $A_n(0)=1$ and

$$A_n(t)=\prod_{j=1}^{[nt]} U_{nj}, \quad A_{nk}=A_n(k/n)$$

$$\Rightarrow \Delta A_{nk}=\alpha_n A_{nk} n^{-1}+\sigma_n A_{nk}\Delta X_{nk}.$$

(A_n will be central in the proof of Proposition 4.2.2).

With $\xi_{nk}=(\tilde{S}_{nk}, A_{nk})^T, \xi_n(t)=\xi_{n[nt]}$, we obtain

$$\Delta\xi_{nk}=a_{nk}\Delta t_{nk}+\beta_{nk}\Delta\psi_{nk}$$

where $\Delta t_{nk}=n^{-1}, a_{nk}=(\mu+\alpha_n\tilde{S}_{nk}, \alpha_n A_{nk})^T, \psi_{nk}=(X_{nk}, Y_{nk})^T$ and

$$\beta_{nk}=\begin{pmatrix}\sigma_n\tilde{S}_{nk} & \varrho\\ \sigma_n A_{nk} & 0\end{pmatrix}.$$

In the notation of *GS* $(x=(x_1, x_2)^T\in\mathbf{R}^2)$

$$a_{nk}(x)=\begin{pmatrix}\alpha_n x_1+\mu\\ \alpha_n x_2\end{pmatrix}, \quad b_{nk}(x)=\begin{pmatrix}\sigma_n x_1 & \varrho\\ \sigma_n x_2 & 0\end{pmatrix}.$$

Since α_n and σ_n have finite limits (see below), the conditions of Theorem 2, p. 190 of *GS*, are satisfied, implying that the measures associated with $\{\xi_n, n\geq 1\}$ are weakly compact. In order to apply Theorem 13, p. 208 of *GS*, it only remains to prove that the limits of α_n and σ_n are as stated in (4.4) to (4.8), and that $\{(X_n, Y_n), n\geq 1\}$ converge to two-dimensional Brownian motion.

Case A. Here $\alpha_n\equiv E(U_{11}-1), \sigma_n\equiv\sigma=(\operatorname{Var} U_{11})^{1/2}$ and the result is a consequence of Donsker's Theorem applied to (X_n, Y_n).

Case B. Let $\gamma=E\log U_{11}$ and $G=\log U_{11}-\gamma$. Then

$$\begin{aligned}\alpha_n &= nE\{\exp[n^{-1}\gamma+n^{-1/2}G]-1\}\\ &= n\{\exp(n^{-1}\gamma)[1+(1/2n)\operatorname{Var} G+o(n^{-1})]-1\}\\ &\to \alpha=\gamma+(1/2)\operatorname{Var} G \quad \text{as } n\to\infty.\end{aligned}$$

Similarly

$$\begin{aligned}\sigma_n^2 &= n\{E\exp(2n^{-1}\gamma+2n^{-1/2}G)-[E\exp(n^{-1}\gamma+n^{-1/2}G)]^2\}\\ &= n\exp(2n^{-1}\gamma)[(1+2n^{-1}\operatorname{Var} G+o(n^{-1}))-(1+(1/2)n^{-1}\operatorname{Var} G+o(n^{-1}))^2]\\ &\to \sigma^2=\operatorname{Var} G \quad \text{as } n\to\infty.\end{aligned}$$

Clearly $\{Y_n(t)=Y_{n[nt]}; t\geq 0\}$ converge weakly to a Brownian motion as $n\to\infty$; it only remains to prove the same thing for $\{X_n(t)=X_{n[nt]}; t\geq 0\}$. This will be done by checking Lindeberg's condition

$$nE(U_{n1}-EU_{n1})^2 1_{\mathscr{E}_n} \to 0 \quad \text{as } n\to\infty$$

where $\mathscr{E}_n=\{|U_{n1}-EU_{n1}|\geq\varepsilon\}$.

In the above expressions U_{n1} will be replaced with $\exp(n^{-1}\gamma+n^{-1/2}G)$.

Lemma A.1. *For* $0<t\leq 1/2$ *and* $y\in\mathbf{R}$

$$[(e^{ty}-1)/t]^2\leq y^2+2e^{2y}.$$

Proof. If $g(t)=e^{ty}$, then

$$\begin{aligned} g(t) &= g(0)+tg'(\zeta(t)), 0\leq\zeta(t)\leq t \\ &= 1+ty\exp[\zeta(t)y] \end{aligned}$$

which implies

$$[(e^{ty}-1)/t]^2 = y^2\exp[2\zeta(t)y] \tag{a.1}$$

If $y\geq 0$, then $y^2<2[1+y+y^2/2\ldots]=2e^y$ and the r.h.s. of (a.1) is smaller than $2e^{2y}$. If $y<0$, $\exp[2\zeta(t)y]\leq 1$ and then the r.h.s. of (a.1) is at most y^2. □

From the lemma

$$\begin{aligned} n(e^{n^{-1/2}G}-Ee^{n^{-1/2}G})^2 &\leq 2n(e^{n^{-1/2}G}-1)^2+2n(1-Ee^{n^{-1/2}G})^2 \\ &\leq 2G^2+4e^{2G}+\sup_n 2n(1-Ee^{n^{-1/2}G})^2 \end{aligned}$$

for $n\geq 4$. We see that

$$\sup_n n(e^{n^{-1}\gamma+n^{-1/2}G}-Ee^{n^{-1}\gamma+n^{-1/2}G})^2$$

is an integrable random variable. Lindeberg's condition results from $\mathscr{E}_n\downarrow\varnothing$ as $n\to\infty$.

Finally, Theorem 13 p. 208 of *GS* tells us that $\{P_n\xi_n^{-1}, n\geq 1\}$ has only one weak limit, the measure on $D[0,T]$ associated with the unique solution of the Itô *SDE*

$$d\xi(t) = \begin{pmatrix}\alpha\xi_1(t)+\mu\\ \alpha\xi_2(t)\end{pmatrix}dt+\begin{pmatrix}\sigma\xi_1(t) & \varrho\\ \sigma\xi_2(t) & 0\end{pmatrix}\begin{pmatrix}dX_t\\ dY_t\end{pmatrix}.$$

This proves (4.9). Eq. (4.3) can be checked using Itô's formula, e.g. Arnold (1974), pp. 90–91. □

Proof of Proposition 4.2.2. Clearly

$$Z_n(t) = \tilde{S}_n(t)/A_n(t)$$

that is to say $Z_n=\xi_{1n}/\xi_{2n}$. The proposition follows from the Continuous Mapping Theorem (Billingsley, 1968, p. 30). □

Proof of Proposition 4.2.3. Define

$$B_n(t) = \sum_{j=0}^{[nt]-1} C_{nj} V_{nj+1} \dots V_{n[nt]}.$$

Then (see (2.8)) $B_n(t) \overset{d}{=} Z_n(t)$. Under assumptions I, II and III B, Propositions 4.2.1 and 4.2.2 imply

$$B_n(t) \overset{d}{\to} \int_0^t e^{-\tilde{X}_t+\tilde{X}_s} d\tilde{Y}_s, Z_n(t) \overset{d}{\to} \int_0^t e^{-\tilde{X}_s} d\tilde{Y}_s. \qquad \square$$

Proof of Proposition 4.2.4. Consider (4.9) and (4.10). The unique solutions to these equations are diffusions with infinitesimal mean and variance

$$a(x) = \alpha x + \mu, \ b(x) = \sigma^2 x^2 + \varrho^2; \quad x \in \mathbf{R}.$$

Therefore, given the same initial conditions, S and Q have the same distribution as random elements of $C[0, \infty)$ (Arnold, 1974, pp. 154–156). $\square$

References

Abramowitz, M. & Stegun, I. A. (1965). *Handbook of mathematical functions*. Dover, New York.

Arnold, L. (1974). *Stochastic differential equations: Theory and Applications*. Wiley, New York.

Billingsley, P. (1968). *Convergence of probability measures*. Wiley, New York.

Boogaert, P., Haezendonck, J. & Delbaen, F. (1988). Limit theorems for the present value of the surplus of an insurance portfolio. *Insurance: Mathematics and Economics* 7, 131–138.

Bourguignon, F. (1974). A particular class of continuous-time stochastic growth models. *J. Econ. Theory* **9**, 141–158.

Bowers, N. L., Hickman, J. C. & Nesbitt, C. J. (1976). Introduction to the dynamics of pension funding. *Transactions of the Society of Actuaries* **28**, 177–203.

Bowers, N. L., Hickman, J. C. & Nesbitt, C. J. (1979). The dynamics of pension funding: Contribution theory. *Transactions of the Society of Actuaries* **31**, 93–119.

Bowers, N. L., Hickman, J. C. and Nesbitt, C. J. (1982). Notes on the dynamics of pension funding. *Insurance: Mathematics and Economics* **4**, 261–270.

Boyle, P. P. (1976). Rates of return as random variables. *Journal of Risk and Insurance* **43**, 693–713.

Brandt, A. (1986). The stochastic equation $Y_{n+1}=A_n Y_n+B_n$ with stationary coefficients. *Adv. Appl. Prob.* **18**, 211–220.

Braun, H. (1986). Weak convergence of assets processes with stochastic interest return. *Scand. Actuarial J.*, 98–106.

Chamayou, J.-F. & Letac, G. (1991). Explicit stationary distributions for compositions of random functions and products of random matrices. *Journal of Theoretical Probability* **4**, 3–35.

Darling, D. A. & Siegert, A. J. F. (1953). The first passage problem for a continuous Markov process. *Ann. Math. Stat.* **24**, 624–639.

Dufresne, D. (1986). Pension funding and random rates of return. In: *Insurance and risk theory* (ed. M. Goovaerts et al.). Reidel, Dordrecht.

Dufresne, D. (1988*a*). Comparison of funding methods in a static environment. *Transactions of the Twenty-Third International Congress of Actuaries*, Helsinki, **2**, 99–114.

Dufresne, D. (1988*b*). Moments of pension contributions and fund levels when rates of return are random. *Journal of the Institute of Actuaries* **115**, 535–544.

Dufresne, D. (1989). Weak convergence of random growth processes with applications to insurance. *Insurance: Mathematics and Economics* **8**, 187–201.

Emmanuel, D. C., Harrison, J. M. & Taylor, A. J. (1975). A diffusion approximation for the ruin function of a risk process with compounding assets. *Scand. Actuarial J.*, 240–247.

Feller, W. (1968). *An introduction to probability theory and its applications*. Vol. 1, 3rd ed. Wiley, New York.

Feller, W. (1971). *An introduction to probability theory and its applications*. Vol. 2, 2nd ed. Wiley, New York.

Garrido, J. (1988). Diffusion premiums for claim severities subject to inflation. *Insurance: Mathematics and Economics* **7**, 123–129.

Gerber, H. (1979). *An introduction to mathematical risk theory*. S. S. Huebner Foundation, University of Pennsylvania, Philadelphia.

Gihman, I. I. & Skorohod, A. V. (1972). *Stochastic differential equations*. Springer-Verlag, New York.

Gihman, I. I. & Skorohod, A. V. (1979). *The theory of stochastic processes*. Vol. 3. Springer-Verlag, New York.

Goldberg, S. (1986). *Introduction to difference equations*. Dover, New York.

Grandell, J. (1977). A class of approximations of ruin probabilities. *Scand. Actuarial J.*, 37–52.

Grandell, J. (1978). A remark on "A class of approximation of ruin probabilities". *Scand. Actuarial J.*, 77–78.

Guess, H. A. & Gillespie, J. H. (1977). Diffusion approximations to linear stochastic difference equations with stationary coefficients. *J. Appl. Prob.* **14**, 58–74.

Harrison, J. M. (1977). Ruin problems with compounding assets. *Stochastic Processes and their Applications* **5**, 67–79.

Hogg, R. V. and Tanis, E. A. (1988). *Probability and statistical inference*. Macmillan, New York.

Iglehart, D. L. (1969). Diffusion approximations in collective risk theory. *J. Appl. Prob.* **6**, 285–292.

Karlin, S. & Taylor, H. M. (1975). *A first course in stochastic processes*. Academic Press, New York.

Karlin, S. & Taylor, H. M. (1981). *A second course in stochastic processes*. Academic Press, New York.

Lassner, F. (1974). Sur certains types de mécanismes additifs en économie stochastique. *C. R. Acad. Sc. Paris Série A* **279**, 33–36.

Lebedev, N. N. (1972). *Special functions and their applications*. Dover, New York.

Liberman, U. (1985). An order statistic characterization of the Poisson renewal process. *Journal of Applied Probability* **22**, 717–722.

Loève, M. (1977) *Probability theory I*. 4th ed. Springer-Verlag, New York.

Loève, M. (1978). *Probability theory II*. 4th ed. Springer-Verlag, New York.

Mandl, P. (1968). *Analytical treatment of one-dimensional Markov process*. Springer-Verlag, Berlin.

Merton, R. C. (1975). An asymptotic theory of growth under uncertainty. *Review of Economic Studies* **42**, 375–393.

O'Brien, T. (1986). A stochastic dynamic approach to pension funding. *Insurance: Mathematics and Economics* **5**, 141–146.

O'Brien, T. (1987). A two-parameter family of pension contribution functions and stochastic optimization. *Insurance: Mathematics and Economics* **6**, 129–134.

Pollard, D. (1984). *Convergence of stochastic processes*. Springer-Verlag, New York.

Ross, S. M. (1983). *Stochastic processes*. Wiley, New York.

Ruohonen, M. (1980). On the probability of ruin of risk processes approximated by a diffusion process. *Scand. Actuarial J.*, 113–120.

Stratonovich, R. L. (1966). A new representation for stochastic integrals and equations. *SIAM J. Control* **4**, 362–371.

Szyszkowski, I. (1987). Invariance principle for integral type functionals of square-integrable martingales. *Stoch. Proc. Appl.* **24**, 269–277.

Takacs, L. (1954). On secondary processes generated by a Poisson process and their application in physics. *Acta Math. Acad. Sci. Hung.* **5**, 203–235.

Takacs, L. (1955). On stochastic processes connected with certain physical recording apparatus. *Acta Math. Acad. Sci. Hung.* **6**, 363–380.

Takacs, L. (1956). On secondary stochastic processes generated by recurrent processes. *Acta Math. Acad. Sci. Hung.* **7**, 19–29.

Taylor, G. C. (1979). Probability of ruin under inflationary conditions or under experience rating. *ASTIN Bulletin* **10**, 149–162.

Taylor, J. R. (1967). The generalized family of aggregate cost methods for pension funding. *Transactions of the Society of Actuaries* **19**, 1–12.

Treuil, P. (1981). Fund development of an earnings related social plan under stabilized conditions. *Transactions of the Society of Actuaries* **33**, 231–246.

Trowbridge, C. L. (1952). Fundamentals of pension funding. *Transactions of the Society of Actuaries* **4**, 17–43.

Trowbridge, C. L. (1963). The unfunded present value family of pension funding methods. *Transactions of the Society of Actuaries* **15**, 151–169.

Vervaat, W. (1972). *Success epochs in Bernoulli trials (with applications in number of theory).* Mathematisch Centrum, Amsterdam.

Vervaat, W. (1979). On a stochastic difference equation and a representation of non-negative infinitely divisible random variables. *Adv. Appl. Prob.* **11**, 750–783.

Waters, H. R. (1983). Probability of ruin for a risk process with claims cost inflation. *Scand. Actuarial J.*, 148–164.

Willmot, G. E. (1989). The total claims distribution under inflationary conditions. *Scand. Actuarial J.*, 1–12.

Winklevoss, H. E. (1977). *Pension mathematics with numerical illustrations.* Irwin, Homewood, Ill.

Wolfe, S. J. (1982). On a continuous analogue of the stochastic difference equations $X_n = \varrho X_{n-1} + B_n$. *Stochastic Processes and their Applications* **12**, 301–312.

Received February 1990

Daniel Dufresne
Département de mathématiques et de statistique
Université de Montréal
P.O. Box 6128, Station A
Montréal, (Québec)
Canada
H3C 3J7