BETA PRODUCTS WITH COMPLEX PARAMETERS

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Abstract

The family consisting of the distributions of products of two independent beta variables is extended to include cases where some of the parameters are complex. This "beta product" distribution is expressible as a Meijer G function. An example (from risk theory) where such a distribution arises is given: an infinite sum of products of independent random variables is shown to have a distribution that is the same as that of the product of a complex-parameter beta product and an independent exponential. The distribution of the infinite sum is a new explicit solution of the stochastic equation $X =$ (in law) $B(X + C)$. Characterisations of some G distributions are also proved.

Keywords: Beta product distribution; Stochastic difference equations; Meijer G functions; G distributions

1. Introduction

The distribution of the product of two independent random variables each with a beta distribution will be extended to cases where some of the parameters are complex. Although this distribution clearly falls within the class of distributions that may be expressed as a G function (= generalized hypergeometric function), it appears that the properties of this distribution have not been studied before; in particular, the possibility of complex parameters seems not to have been noticed. In order to show that complex parameters are not just a useless curiosity, an example of a variable possessing such a distribution will be given, originating in risk theory; this example is also a new explicit solution the equation $X \overset{\mathrm{d}}{=} B(X + C)$ ("$\overset{\mathrm{d}}{=}$" will mean "has the same distribution as").

These questions are related to the study of probability distributions on $\mathbb{R}_+$ with a Mellin transform expressible as

$$\frac{\Gamma(a_1+t)\cdots\Gamma(a_p+t)}{\Gamma(a_1)\cdots\Gamma(a_p)} \times \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(b_1+t)\cdots\Gamma(b_q+t)} \times \frac{\Gamma(c_1-t)\cdots\Gamma(c_r-t)}{\Gamma(c_1)\cdots\Gamma(c_r)} \times \frac{\Gamma(d_1)\cdots\Gamma(d_s)}{\Gamma(d_1-t)\cdots\Gamma(d_s-t)} \qquad (1.1)$$

for constants $\{a_j, b_k, c_\ell, d_m\}$. The probability distribution with this Mellin transform, when it exists, was denoted

$$\left(\begin{matrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_q \end{matrix} \,\middle|\, \begin{matrix} c_1 & \cdots & c_r \\ d_1 & \cdots & d_s \end{matrix}\right)$$

in Dufresne (1998); when there are no c's or d's this is shortened to

$$\begin{pmatrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_q \end{pmatrix} := \left(\begin{matrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_q \end{matrix} \;\middle|\; \begin{matrix} - \\ - \end{matrix}\right).$$

This family of distributions has been studied by others, see Mathai & Saxena (1973) for references. I will refer to them as "G distributions", though this is not a common expression. This name will be used because Meijer's G functions are defined as

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} \;\middle|\; z\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_j - s)\prod_{j=1}^n \Gamma(1-a_j+s)}{\prod_{j=m+1}^q \Gamma(1-b_j+s)\prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds,$$

and this says that distributions with a Mellin transform as in (1.1) have a density which is a constant times a G function. Beta and gamma distributions and their multiplicative convolutions are members of this family.

The usual beta density function cannot have one or both parameters in $\mathbb{C}-\mathbb{R}$, but the multiplicative convolution of two beta densities with some parameters complex, or even negative, does make sense as a probability density, and the result is a new probability distribution, different from all real-parameter beta product distributions. The density function of this complex-parameter beta product explicitly depends on those complex parameters. This must be one of the rare distribution families which has "natural" complex parameters.

The definition and some of the properties of the beta product distribution are given in Section 2. The set of possible parameters is completely specified, and this allows the characterisation of all G distributions $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ on $(0,1)$. An example where the beta product distribution arises is given in Section 3; the example is set in the context of risk theory, but mathematically it is essentially about an infinite sum of products of random variables, the distribution of which satisfies the functional equation $X \overset{\mathrm{d}}{=} B(X+C)$. Sections 4 and 5 show that there is no extension to complex parameters when considering the multiplicative convolutions of two gamma densities, or multiplicative convolutions of a gamma and a beta density.

A number of authors have studied the distribution of the product of beta variables, in particular because this topic relates to the distribution of Wilks' statistic. Springer & Thompson (1970) express the densities of products of beta and gamma variables in terms of G functions, while Bhargava & Khatri (1981) relate products of beta variables to the logbeta distribution (though they do not call it that way), as we do at the end of Section 2; Nadarajah (2005) studies sums, differences, products and ratios of non-central beta distributions, and expresses the resulting densities in terms of hypergeometric functions, as will be the case in this paper, though we only consider the ordinary beta distribution, not the non-central forms. Other references on products of beta variables include Jambunathan (1954), Tang & Gupta (1984), Fan (1991) and Pham-Gia (2000); none of those extend the parameter set to include complex or negative numbers as is done in this paper.

Classic references on the stochastic difference equation $X =$ (in law) $B(X+C)$ (which appears in Section 3) are Kesten (1973), Vervaat (1979) and Chamayou & Letac (1991);

see also Dufresne (1996, 1998). A reference for the special functions appearing in this paper is Lebedev (1972).

The usual shifted factorials

$$(a)_0 = 1, \qquad (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad n \geq 1,$$

will be used. The Mellin transform of the distribution of a non-negative variable X is the function $r \mapsto \mathsf{E}X^r$. If X has a density f then this is

$$\mathsf{E}X^r = \int_0^\infty dx\, x^r f(x).$$

For consistency, the Mellin transform $\mathcal{M}f(r)$ of any function $f : \mathbb{R}_+ \to \mathbb{R}$ will be defined by the same integral. (*N.B.* In analysis the Mellin transform is usually defined as $\int dx x^{r-1} f(x)$.) The multiplicative convolution of functions f, g on $(0, \infty)$ is defined as

$$(f \odot g)(y) = \int_0^\infty \frac{dx}{x} f(x) g\left(\frac{y}{x}\right).$$

Hence, the density of the product $Y = X_1 X_2$ of two independent positive variables with densities f_1, f_2 is $f_1 \odot f_2$. We recall the beta and gamma functions:

$$\int_0^1 du\, u^{a-1}(1-u)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = B(a,b), \qquad \int_0^\infty dx\, x^{c-1} e^{-x} = \Gamma(c),$$

which are well-defined for $\mathsf{Re}(a), \mathsf{Re}(b), \mathsf{Re}(c) > 0$. We also define

$$f_{a,b}(u) = \frac{1}{B(a,b)} u^{a-1}(1-u)^{b-1}\mathbf{1}_{\{0<u<1\}}$$
$$h_c(x) = \frac{x^{c-1}e^{-x}}{\Gamma(c)}$$

for $\mathsf{Re}(a), \mathsf{Re}(b), \mathsf{Re}(c) > 0$, with Mellin transforms

$$\int_0^\infty du\, u^r f_{a,b}(u) = \frac{\Gamma(a+r)\Gamma(a+b)}{\Gamma(a+b+r)\Gamma(a)}, \qquad \int_0^\infty dx\, x^r h_c(x) = \frac{\Gamma(c+r)}{\Gamma(c)}$$

(for $\mathsf{Re}(r) > -a$ and $\mathsf{Re}(r) > -c$, respectively). When $a, b, c > 0$, the functions $f_{a,b}$ and h_c are the density functions of the usual **Beta**(a, b) and **Gamma**(c) distributions.

The Gauss hypergeometric function ${}_2F_1$ has an analytic continuation to other regions of the complex plane, but for our purposes it will be sufficient to think of it as

$${}_2F_1(a,b;c;z) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \qquad c \neq 0, -1, -2, \ldots, \qquad |z| < 1. \tag{1.2}$$

The parameters a, b, c may be complex. A probabilistic interpretation of this function is as follows. If $B \sim$ **Beta**(a, b) is independent of $G \sim$ **Gamma**(c), then, calculating $\mathsf{E}e^{zBG}$ in two different ways (summing the moments or conditioning on B), we get, for $a,b,c>0$, $|z|<1$,

$$ {}_2F_1(a,c;a+b;z) \;=\; \frac{1}{B(a,b)}\int_0^1 du\, u^{a-1}(1-u)^{b-1}(1-uz)^{-c}. \qquad (1.3) $$

Dropping the probabilistic interpretation, this formula (due to Euler) extends to $\mathsf{Re}(a)>0$, $\mathsf{Re}(b)>0$, $c\in\mathbb{C}$, $|\arg(1-z)|<\pi$ (Lebedev, 1972, p.240).

2. The beta product distribution with complex parameters

The function $f_{a,b}$ is not a probability density for complex values of the parameters a and b, but it will now be shown that there are choices of complex numbers a,b,c,d such that $f_{a,b}\odot f_{c,d}$ is a true probability density function. To ensure that the integrals below converge, suppose $\mathsf{Re}(a),\mathsf{Re}(b),\mathsf{Re}(c),\mathsf{Re}(d)>0$; then the product convolution is, for $0<u<1$:

$$
\begin{aligned}
g(u) &= \int_u^1 \frac{dv}{v} f_{a,b}\Big(\frac{u}{v}\Big) f_{c,d}(v)\\
&= \frac{1}{B(a,b)B(c,d)}\int_0^{1-u}\frac{dw}{1-w} f_{a,b}\Big(\frac{u}{1-w}\Big) f_{c,d}(1-w)\\
&= \frac{1-u}{B(a,b)B(c,d)}\int_0^1 \frac{dx}{1-x(1-u)} f_{a,b}\Big(\frac{u}{1-x(1-u)}\Big) f_{c,d}(1-x(1-u))\\
&= \frac{u^{a-1}(1-u)^{b+d-1}}{B(a,b)B(c,d)}\int_0^1 dx\, x^{d-1}(1-x)^{b-1}(1-x(1-u))^{c-a-b}\\
&= \frac{\Gamma(a+b)\Gamma(c+d)}{\Gamma(a)\Gamma(c)\Gamma(b+d)}\, u^{a-1}(1-u)^{b+d-1}\,{}_2F_1(a+b-c,d;b+d;1-u)
\end{aligned}
$$

by (1.3).

Remark. Since multiplicative convolution is symmetrical, the roles of (a,b) and (c,d) may be reversed in the previous derivation, giving

$$ g(u) \;=\; \frac{\Gamma(a+b)\Gamma(c+d)}{\Gamma(a)\Gamma(c)\Gamma(b+d)}\, u^{c-1}(1-u)^{b+d-1}\,{}_2F_1(c+b-a,b;b+d;1-u). $$

Equating the two expressions for $g(u)$ and letting $z=1-u$, $\alpha=a+b-c$, $\beta=d$, $\gamma=b+d$, we obtain Euler's formula (Lebedev, 1972, p.248):

$$ {}_2F_1(\alpha,\beta;\gamma;z) \;=\; (1-z)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z). $$ □

Define

$$ g_{a,b,c,d}(u) \;=\; \frac{\Gamma(a+b)\Gamma(c+d)}{\Gamma(a)\Gamma(c)\Gamma(b+d)}\, u^{a-1}(1-u)^{b+d-1}\,{}_2F_1(a+b-c,d;b+d;1-u)\mathbf{1}_{\{0<u<1\}}. \qquad (2.1) $$

We look for values of a,b,c,d (besides $a,b,c,d>0$) for which the expression in (2.1) is a probability density function. First, when $u\uparrow 1$ the hypergeometric function tends to 1, and thus

$$ g_{a,b,c,d}(u) \;\sim\; \frac{\Gamma(a+b)\Gamma(c+d)}{\Gamma(a)\Gamma(c)\Gamma(b+d)}\,(1-u)^{b+d-1} $$

as $u \uparrow 1$. If $\mathrm{Im}(b+d) = \gamma \neq 0$, then $(1-u)^{\mathrm{Im}(b+d)} = \exp(i\gamma \log(1-u))$ goes round the unit circle in $\mathbb{C}$, and thus has real and imaginary parts that change sign as u increases to 1. This implies that, irrespective of the rest of (2.1), as u tends to 1 from the left the function $g_{a,b,c,d}(u)$ cannot remain real and non-negative. Therefore we must have $\mathrm{Im}(b+d) = 0$. In order for the density to be integrable near 0, it is also required that $\mathsf{Re}(b+d) > 0$.

Under the assumption that $\mathsf{Re}(a), \mathsf{Re}(b), \mathsf{Re}(c), \mathsf{Re}(d) > 0$, the convolution property of Mellin transforms yields

$$\int_0^1 du\, u^r g_{a,b,c,d}(u) \;=\; \frac{\Gamma(a+r)\Gamma(c+r)\Gamma(a+b)\Gamma(c+d)}{\Gamma(a+b+r)\Gamma(c+d+r)\Gamma(a)\Gamma(c)} \tag{2.2}$$

for $\mathrm{Re}(r) > -\min(\mathsf{Re}(a), \mathsf{Re}(c))$. Either side is an analytic function of a, b, c, d, r in the indicated region, but the restrictions $\mathsf{Re}(b), \mathsf{Re}(d) > 0$ may be removed by analytic continuation. More precisely, if $\mathsf{Re}(a), \mathsf{Re}(c), \mathsf{Re}(a+b), \mathsf{Re}(c+d), b+d > 0$, then $g_{a,b,c,d}(u)$ will now be shown to be integrable over $(0,1)$, and so the left-hand side of (2.2) will be an analytic function or r, as is the right-hand side. This implies equality of both sides in the larger region.

To prove that $g_{a,b,c,d}$ is integrable over $(0,1)$ under these weaker conditions, recall that $b+d > 0$ and ${}_2F_1(\alpha, \beta; \gamma; 0) = 1$ imply that $\int_{1-\epsilon}^1 |g_{a,b,c,d}(u)|\, du$ is finite. Next, assume that $\mathsf{Re}(c-a) > 0$. Then, using $\mathsf{Re}(a) > 0$ and the formula (Lebedev, 1972, p.244)

$$\lim_{z \to 1-} {}_2F_1(\alpha, \beta; \gamma; z) \;=\; \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \qquad \text{if } \mathrm{Re}(\gamma-\alpha-\beta) > 0, \tag{2.3}$$

we see that $\int_0^\epsilon |g(u)|\, du$ is finite. If $\mathsf{Re}(c-a) < 0$, then reverse the roles of (a,b) and (c,d) in the previous argument to get the same result. If $\mathsf{Re}(c-a) = 0, c-a \neq 0$, then use the formula (Lebedev, 1972, p.249):

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; z) \;=\;& \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\, {}_2F_1(\alpha, \beta; 1+\alpha+\beta-\gamma; 1-z) \\ &+ (1-z)^{\gamma-\alpha-\beta} \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\, {}_2F_1(\gamma-\alpha, \gamma-\beta; 1-\alpha-\beta+\gamma; 1-z), \\ & |\arg z| < \pi,\ |\arg(1-z)| < \pi,\ \alpha+\beta-\gamma \neq 0, \pm 1, \pm 2 \ldots, \end{aligned} \tag{2.4}$$

to rewrite (2.1) as the sum of two functions integrable over $(0, \epsilon)$.

Finally, if $c = a$, then the hypergeometric function in (2.1) is of the form ${}_2F_1(\alpha, \beta; \alpha + \beta; z)$ with $\gamma - \alpha - \beta = 0$, so the above arguments break down. If $b, d > 0$ there is nothing to prove, as we are then back to the original derivation of (2.2). Suppose $\mathrm{Im}(b) \neq 0$. From the assumption that $b+d > 0$ this means that $b = b_1 + ib_2$, $d = d_1 - ib_2$ with $b_1 + d_1 > 0, b_2 \neq 0$. The hypergeometric function in (2.1) now equals

$$\sum_{n=0}^{\infty} \frac{(b)_n (d)_n}{(b+d)_n} \frac{(1-u)^n}{n!}.$$

Compare this sum with the one obtained for $b' = b_1' + ib_2$, $d' = d_1' - ib_2$, with $b_1' = d_1' = (b_1 + d_1)/2$. By squaring and expanding both sides, it can be verified that

$$|(b+n)(d+n)| \;\leq\; |(b'+n)(d'+n)| \tag{2.5}$$

for all n large enough. It follows that for $u \in (0, \epsilon)$ the norm of (2.1) is bounded above by u^{a-1} times $K + K'\,{}_2F_1(b', d'; b'+d'; 1-u)$, where K_1, K_2 are constants. This is integrable over $(0, \epsilon)$ by what was seen above in the case $\mathsf{Re}(c-a) = 0, c \neq a$. (More details on this step in the Appendix.) We have thus proved that (2.2) holds if $\mathsf{Re}(a), \mathsf{Re}(c), \mathsf{Re}(a+b), \mathsf{Re}(c+d), b+d > 0$ and $\mathsf{Re}(r) > -\min(\mathsf{Re}(a), \mathsf{Re}(c))$.

The theorem below gives sufficient conditions for all terms of the series expansion for the ${}_2F_1$ in (2.1) to be non-negative, thus proving that $g_{a,b,c,d}$ is a probability density.

Theorem 1. *The function $g_{a,b,c,d}$ is a probability density function if $a, c, b+d$ are real and positive, $\mathsf{Re}(a+b), \mathsf{Re}(c+d) > 0$, and either*

(a) (real case) all parameters are real and $\min(a, c) < \min(a+b, c+d)$*; or*

(b) (complex case) $\mathrm{Im}(b) = -\mathrm{Im}(d) \neq 0$ *and* $a + b = \overline{c+d}$.

Proof. By (2.2) we know that $g_{a,b,c,d}(u)$ integrates to 1 in cases (a) and (b); it only remains to check that the function is non-negative. In case (a), if $b, d > 0$ then $g_{a,b,c,d}$ is the multiplicative convolution of two probability densities, and so is a probability density. If $b \leq 0$, then necessarily $d > 0$; moreover, $\min(a, c) < \min(a+b, c+d)$ implies $a + b > c$, since $b \leq 0$ implies $a + b \leq a$. We then write the hypergeometric function in (2.1) as

$$\sum_{n=0}^{\infty} \frac{(a+b-c)_n (d)_n}{(b+d)_n} \frac{(1-u)^n}{n!}, \tag{2.6}$$

the terms of which are all positive. In case $d \leq 0$ just reverse the roles of (a, b) and (c, d) to conclude once more that $g_{a,b,c,d} \geq 0$.

In case (b), the leading constant in (2.1) is real and positive because $\Gamma(z)\Gamma(\overline{z}) = \Gamma(z)\overline{\Gamma(z)} \in \mathbb{R}_{++}$ and so

$$\frac{\Gamma(a+b)\Gamma(c+d)}{\Gamma(a)\Gamma(c)\Gamma(b+d)} \;=\; \frac{|\Gamma(a+b)|^2}{\Gamma(a)\Gamma(c)\Gamma(b+d)} \;>\; 0.$$

The denominator of each term in (2.6) is positive, and so is the numerator , since $a + b = \overline{c+d}$ implies

$$(a+b-c)_n (d)_n \;=\; (a+b-c)_n (\overline{a+b-c})_n \;=\; |(a+b-c)_n|^2.$$

Hence, $g_{a,b,c,d}(u)$ is positive for $0 < u < 1$, and (2.1) says that it integrates to 1. □

Definition. *The probability distribution with density $g_{a,b,c,d}$ will be called **beta product**, denoted* $\mathbf{BetaP}(a, b, c, d)$.

Eq.(2.2) gives the Mellin transform of $B \sim \mathbf{BetaP}(a, b, c, d)$, and so its moments are

$$\mathsf{E}B^n \;=\; \frac{(a)_n (c)_n}{(a+b)_n (c+d)_n}, \qquad n = 1, 2, \ldots.$$

In the complex case (part (b) of theorem) this is may also be written as

$$\mathsf{E}B^n \;=\; \frac{(a)_n (c)_n}{|(a+b)_n|^2}, \qquad n = 1, 2, \ldots.$$

It is not obvious that those moments are smaller than 1 in cases (a) and (b) of Theorem 1. In case (a), if $b, d > 0$ then all moments are obviously in (0,1), since they are the products of moments of two ordinary beta distributions. If $b \leq 0$, then as seen in the proof we must have $d > 0$ and $a + b > c$, and so

$$\mathsf{E}B \; < \; 1 \;\Leftrightarrow\; (a+b)(c+d) - ac > 0 \;\Leftrightarrow\; ad + bc + bd \; > \; 0.$$

But $ad+bc+bd = (a+b)d+bc > cd+bc = c(b+d) > 0$, which shows $\mathsf{E}B < 1$. It is similarly checked that $\mathsf{E}B^{n+1}/\mathsf{E}B^n < 1$, as it should be. In the complex case, writing $b = b_1 + ib_2$, $d = d_1 - id_2$, we find that $a, c, b + d > 0$, $a + b = \overline{c + d}$ imply

$$\min(a, c) \; < \; \min(a + b_1, c + d_1) \; = \; a + b_1 \; = \; c + d_1,$$

so the same arguments used in case (a) lead to

$$ac \; < \; (a + b_1)(c + d_1) \; = \; (a + b_1)^2 \; \leq \; |a + b|^2,$$

and so to $\mathsf{E}B^{n+1}/\mathsf{E}B^n < 1$ for all $n = 0, 1, \ldots$.

The $\mathbf{BetaP}(a, b, c, d)$ distribution is determined by its moments, and its Laplace transform is a hypergeometric function:

$$\mathsf{E}e^{-sB} \; = \; \sum_{n=0}^{\infty} \frac{(a)_n (c)_n}{(a + b)_n (c + d)_n} \frac{(-s)^n}{n!} \; = \; {}_2F_2(a, c; a + b, c + d; -s), \qquad s \in \mathbb{C}.$$

The restrictions on a, b, c, d allow one of b and d (but not both) to have a negative real part. For instance, the $\mathbf{BetaP}(2, 5, 8, -1)$ density is shown in Figure 1.

[***Figure 1 approximately here***]

Theorem 1 leaves open the question of whether there might be other parameters (a, b, c, d) that are not all real and that also lead to a true probability density function $g_{a,b,c,d}$. Consider the case

$$a = 3 + i/10, \;\; b = 2 + i, \;\; c = 3 - i/10, \;\; d = 2 - i$$

(see Figure 2). This is one instance where $\mathsf{Re}(a) > 0$, $c = \overline{a}$, $b + d > 0$, $a + b = \overline{c + d}$. In those cases the moments

$$\int_0^1 du\, u^n g_{a,b,c,d}(u) \; = \; \frac{|(a)_n|^2}{|(a + b)_n|^2}, \qquad n = 0, 1, 2, \ldots,$$

are all positive, and $g_{a,b,c,d}$ integrates to 1. By all appearances the function in Figure 2 is a probability density function. This leads to a more detailed analysis of the problem, and the following result.

[***Figure 2 approximately here***]

Characterisation of some G distributions

Theorem 2. *Consider the G distributions on* $(0,1)$ *denoted* $\begin{pmatrix} a & c \\ a+b & c+d \end{pmatrix}$, *with all parameters allowed to be complex. Suppose this is a simplified expression, meaning that* $a \neq a+b$, $a \neq c+d$, *and so on. Then this is the* $\mathbf{Beta}(a,b,c,d)$ *distribution, with* a, c *and* $b+d$ *are and positive,* $\mathsf{Re}(a+b), \mathsf{Re}(c+d) > 0$, *and either*

(a) (real case) all parameters are real and $\min(a,c) < \min(a+b, c+d)$*; or*

(b) (complex case) $\mathrm{Im}(b) = -\mathrm{Im}(d) \neq 0$ *and* $a+b = \overline{c+d}$.

The consequence is that the cases listed in Theorem 1 are the only ones where $g_{a,b,c,d}$ *is a probability density function, apart from the "degenerate" cases where one of* a *or* c *equals* $a+b$ *or* $c+d$*, which correspond to an ordinary beta distribution.*

Proof. The notation

$$X \sim \begin{pmatrix} a & c \\ a+b & c+d \end{pmatrix}$$

means that

$$\mathsf{E}X^r = \frac{\Gamma(a+r)\Gamma(c+r)\Gamma(a+b)\Gamma(c+d)}{\Gamma(a+b+r)\Gamma(c+d+r)\Gamma(a)\Gamma(c)} \qquad (2.7)$$

for r in some interval that includes 0. If X only takes values in $(0,1)$, then $\mathsf{E}X^r$ is finite for all $r \geq 0$, and must equal (2.7). The numerator has poles at $r = -a, -c$ (which are not offset by poles of the denominator), and so we must have $\mathsf{Re}(a), \mathsf{Re}(c) > 0$. We thus assume that (2.7) holds for all $r > -\min(\mathsf{Re}(a), \mathsf{Re}(c))$. (More exactly, since $X \in (0,1)$, the function $\mathsf{E}X^r$ must have a pole at the real number r^* where it stops existing, the "abcissa of convergence", see Widder (1941, p.59); hence, if neither a nor c is real, then necessarily $\mathsf{E}X^r$ must exist and be finite for all $r \in \mathbb{R}$.)

Moreover, $\mathsf{E}X^r$ tends to 0 as $r \to \infty$, and we apply the asymptotic result

$$\frac{\Gamma(\alpha+z)}{\Gamma(\beta+z)} \sim z^{\alpha-\beta}\left[1 + \frac{(\alpha-\beta)(\alpha+\beta-1)}{2z} + \mathcal{O}(z^{-2})\right], \qquad |\arg z| < \pi$$

(Lebedev, 1972, p.15), to get

$$\mathsf{E}X^r \sim Kr^{-b-d}$$

(K a constant) as $r \to \infty$, which implies $\mathrm{Im}(b+d) = 0$, $b+d > 0$. From (2.2), this means that X has a density given by (2.1).

If $\mathsf{Re}(c-a) > 0$, then (2.3) leads to

$$g(u) \sim Ku^{a-1} \qquad \text{as } u \downarrow 0$$

(K a constant), and this entails $\mathrm{Im}(a) = 0$, $\mathsf{Re}(a) > 0$. For real r, the ratio

$$\frac{\mathsf{E}X^r}{\mathsf{E}X^{r+1}} = \frac{(a+b+r)(c+d+r)}{(a+r)(c+r)}$$

is real, and $a \in \mathbb{R}$ further implies that

$$\frac{(a+b+r)(c+d+r)}{(c+r)} \;=\; g(r)$$

is also real for all $r \geq 0$. Multiply each side by $c+r$ and differentiate twice to get

$$2 \;=\; g''(r)(c+r) + 2g'(r),$$

which in turn tells us that $c \in \mathbb{R}$, and so $c > 0$. That $g(0)$ is real then says that $(a+b)(c+d) \in \mathbb{R}$. There are two possibilities: (1) both b and d are real, and (2) at least one of b or d is complex.

In case (1), (2.7) has poles at $-a, -c$ and zeros at $-(a+b)$, $-(c+d)$. Since $\mathsf{E}X^r > 0$ for all real r such that the expectation exists, we must have $\min(a,c) < \min(a+b, c+d)$ (*i.e.* a pole must be met before a zero when travelling left on the real axis). We have thus proved that $\mathsf{Re}(c-a) > 0$, $b, d \in \mathbb{R}$ imply case (a) of the theorem.

In case (2), the numbers $z_1 = a+b$, $z_2 = c+d$ are not both real, have a positive product (because $\mathsf{E}X = ac/(a+b)(c+d) > 0$ and $ac > 0$) and a positive sum (since $a, c, b+d > 0$); hence we must have $z_1 = \overline{z}_2$ and $\mathsf{Re}(z_1) = \mathsf{Re}(z_1+z_2)/2 > 0$, that is

$$a+b = \overline{c+d}, \qquad \mathsf{Re}(a+b) = \mathsf{Re}(c+d) > 0. \tag{2.8}$$

We have thus shown that if $\mathsf{Re}(c-a) > 0$ and if one of b and d (or both) is not real, then case (b) of Theorem 2 ensues. If $\mathsf{Re}(a-c) > 0$, then we can reverse the roles of (a,b) and (c,d) in the above, and get the same conclusion, *i.e.* either (a) or (b) holds.

We now turn to the case $\mathsf{Re}(c-a) = 0$, and show that necessarily $a, c > 0$ once again. We first show that we must have $c = \overline{a}$. From (2.7),

$$\frac{\mathsf{E}X^r}{\mathsf{E}X^{r+1}} \;=\; \frac{(a+b+r)(c+d+r)}{(a+r)(c+r)} \;=\; k(r) \;\in\; \mathbb{R} \tag{2.9}$$

for any $r > -\mathsf{Re}(a)$. This is the same as

$$(a+b+r)(c+d+r) \;=\; k(r)(a+r)(c+r).$$

Write $a = a_1 + ia_2$, $c = a_1 + ic_2$ with a_1, a_2, c_2 real, differentiate the last identity twice and then let $r \to -a_1$ to get

$$2 \;=\; k''(-a_1)(-a_2c_2) + 2k'(-a_1)i(a_2+c_2) + 2k(-a_1).$$

The first and last terms on the right are real, and so must the middle term, implying $a_2 + c_2 = 0$, which means $a = \overline{c}$. (The function h and its derivatives at $-a_1$ exist if both a_2 and c_2 are non-zero; if only one is not zero, say c_2, then the same argument works if $k(r)$ is replaced with $\tilde{k}(r) = k(r)(a+r) = (a+b+r)(c+d+r)/(c+r)$.)

Next, we show that $\mathrm{Im}(a) = \mathrm{Im}(c) = 0$. Suppose this is not the case, that is $a_2 = \mathrm{Im}(a) = -\mathrm{Im}(c) \neq 0$. The denominator in (2.9) is real since $a = \overline{c}$, and so must be the numerator, call it $p(r)$. Write $b = b_1 + ib_2$, $d = d_1 - ib_2$. Then

$$p(-a_1 - b_1) \;=\; (ia_2 + ib_2)(-ia_2 + d_1 - b_1 - ib_2)$$

is real, implying either (i) $a_2 = -b_2$ or (ii) $b_1 = d_1$, or both. Case (i) implies $\mathsf{E}X^{-a_1-b_1} = 0$ by (2.7), a contradiction; we therefore exclude (i). In case (ii) we have $b = \overline{d}$. Applying formulas (2.2), (2.1) and (2.4), we then know that the density of X is

$$u^{a_1-1}(1-u)^{b+d-1}[Cu^{ia_2}{}_2F_1(b+2ia_2, d; 1+2ia_2; u) + \overline{C}u^{-ia_2}{}_2F_1(d-2ia_2, b; 1-2ia_2; u))]$$

where C is a constant. The expression in square brackets is the sum of a series $u^{ia_2}\sum \alpha_n u^n$ and its conjugate, and may be expressed as:

$$\sum_{n=0}^{\infty} 2\mathsf{Re}(u^{ia_2}\alpha_n)u^n \;\sim\; 2\mathsf{Re}(u^{ia_2}\alpha_0) \qquad \text{as } u \to 0.$$

The last expression takes positive and negative values as u tends to 0, which is a contradiction, and we conclude that $a_2 = 0$.

We have just proved that $\mathsf{Re}(c-a) = 0$ implies $a = c$. The other claims then easily follow from $a = c > 0$, $b + d > 0$. There are again two possibilities: (1) both b and d are real, and (2) at least one of b or d is complex. In case (1) the requirement that one pole of $\mathsf{E}X^r$ be to the right of all zeros implies $b, d > 0$, and this means that case (a) of the theorem holds. In case (2), (2.8) is obtained from the same arguments as before, implying case (b) of the theorem. When $a = c$ we moreover must have $b = \overline{d}$. □

Theorem 2 says that the function in Figure 2 is NOT a probability density function. It is not immediately clear what the problem might be with that function, but, after looking at the graph more closely (and incresing the workking precison of Mathematica's "NIntegrate" function), one finds that the function takes negative values very close to the origin, see Figure 3. As the last part of the proof of Theorem 2 indicates, the function has damped oscillations as $u \to 0$.

[***Figure 3 approximately here***]

Beta product viewed as discrete combination of ordinary betas

The $\mathbf{BetaP}(a, b, c, d)$ density (2.1) can be expanded as a series, showing that the distribution is a combination of $\mathbf{Beta}(a, b + d + n)$ distributions, with weights

$$w_n \;=\; \frac{\Gamma(a+c)\Gamma(c+d)(a+b-c)_n(d)_n}{\Gamma(c)\Gamma(a+b+d+n)n!}, \qquad n = 0, 1, \ldots.$$

Observe that this expression as a combination of ordinary beta distributions is in general not unique, since another one is found by reversing the roles of (a, b) and (c, d). The weights $\{w_n\}$ are not necessarily positive nor real. It is also possible that only a finite number of those weights are non-zero, this happens when either $a + b - c, b, c + d - a$ or d is a non-positive integer (which causes the hypergeometric function to terminate).

Connection with the logbeta distribution

Theorem 1 may be rewritten in terms of the logbeta distribution. In Dufresne (2007), this distribution is defined as follows: for any $\alpha, \beta, \gamma > 0$,

$$X \sim \mathbf{LogBeta}(\alpha, \beta, \gamma) \quad \Leftrightarrow \quad e^{-\gamma X} \sim \mathbf{Beta}(\beta/\gamma, \alpha)$$

(γ is a scale parameter). The density of this distribution is

$$\frac{\gamma}{B\big(\frac{\beta}{\gamma},\alpha\big)}e^{-\beta x}(1-e^{-\gamma x})^{\alpha-1}\mathbf{1}_{\{x>0\}}. \tag{2.10}$$

This family includes the exponential ($\alpha=1$), and gamma distributions are limit points of the logbeta family.

What was said about multiplicative convolutions of beta densities translates immediately into results about additive convolutions of logbeta densities. Theorems 1 and 2 then say that two such functions with parameters $(\alpha_j,\beta_j,\gamma)$, $j=1,2$, have an additive convolution which is a probability density if, and only if, $\gamma,\beta_1,\beta_2,\alpha_1+\alpha_2,\mathsf{Re}\big(\frac{\beta_1}{\gamma}+\alpha_1\big),\mathsf{Re}\big(\frac{\beta_2}{\gamma}+\alpha_2\big)>0$ and either (a$'$) all parameters are real and $\min(\beta_1,\beta_2)<\min(\beta_1+\gamma\alpha_1,\beta_2+\gamma\alpha_2)$, or (b$'$) α_1,α_2 are complex and $\beta_1+\gamma\alpha_1=\beta_2+\gamma\overline{\alpha}_2$.

The $\mathbf{BetaP}(a,b,c,d)$ *is a new distribution for* b *or* $d\notin\mathbb{R}_{++}$

It is important to check that the $\mathbf{BetaP}(a,b,c,d)$ is in fact a new distribution when b,d are not both positive and real. It could turn out to be the same as the multiplicative convolution of some ordinary real-parameter beta distributions. We now show that if $b\notin\mathbb{R}_{++}$ then it is not possible to find positive numbers a_1,a_2,b_1,b_2 such that $\mathbf{Beta}(a,b,c,d)=\mathbf{Beta}(a_1,b_1)\odot\mathbf{BetaP}(a_2,b_2)$. This is easily done by proving that Mellin transforms cannot be equal. If they were, then taking the ratio of the transforms at $r+1$ and at r, we would get

$$\frac{(a+r)(c+r)}{(a+b+r)(c+d+r)} \;=\; \frac{(a_1+r)(a_2+r)}{(a_1+b_1+r)(a_2+b_2+r)}$$

for any r with $\mathsf{Re}(r)>-\min(a,c)$. Both are meromorphic functions, and thus must have the same zeros and the same poles; this is impossible unless there is equality between the quadruples (a,b,c,d) and (a_1,b_1,a_2,b_2) (or (a,b,c,d) and (a_2,b_2,a_1,b_1))

Factorisations

Mellin transforms immediately show that for any $a,b,c>0$,

$$\mathbf{Beta}(a,b)\odot\mathbf{Beta}(a+b,c) \;=\; \mathbf{Beta}(a,b+c).$$

This trivially implies that when $a_1,a_2,b_1,b_2,c_1,c_2>0$ we also have

$$\mathbf{BetaP}(a_1,b_1,a_2,b_2)\odot\mathbf{BetaP}(a_1+b_1,c_1,a_2+b_2,c_2) \;=\; \mathbf{BetaP}(a_1,b_1+c_1,a_2,b_2+c_2).$$

When b_1 (and therefore b_2) is not real, however, the last identity does not hold, because it involves a first parameter a_1+b_1 that is not real. There is another multiplicative property of the **BetaP** distribution which does hold in real and complex cases. Suppose that $a<\gamma<a+\mathsf{Re}(b)$; then Mellin transforms imply that

$$\mathbf{BetaP}(a,b,c,d) \;=\; \mathbf{BetaP}(\gamma,a+b-\gamma,c,d)\odot\mathbf{Beta}(a,\gamma-a).$$

In particular, when $\mathrm{Im}(b)\neq 0$ and $a<c<a+\mathsf{Re}(b)$,

$$\mathbf{BetaP}(a,b,c,d) \;=\; \mathbf{BetaP}(c,a+b-c,c,a+\overline{b}-c)\odot\mathbf{Beta}(a,c-a).$$

3. An example

Consider a model under which i.i.d. claims $\{C_n\}$ occur at times $\{T_n\}$, and that one wishes to find the law of the discounted value of all future claims. If the discount rate is $r > 0$, then this is

$$X = \sum_{n=1}^{\infty} e^{-rT_n} C_n.$$

Let the waiting times $W_1 = T_1, W_n = T_n - T_{n-1}, n \geq 2$ be i.i.d., making $\{T_n\}$ a renewal process, and assume moreover that $\{T_n\}$ and $\{C_n\}$ are independent. Then the above sum may be rewritten as

$$X = \sum_{n=1}^{\infty} B_1 \cdots B_n C_n,$$

if $B_n = e^{-rW_n}$. Such sums of products of random variables occur in a variety of models and have been sudied for a while. It is known (see Vervaat, 1979 for details) that X is finite w.p.1 if $\mathsf{P}(W_1 > 0) > 0$ and $\mathsf{E}\log|C_1| < \infty$, and that it is then the unique solution of the identity in law

$$X \stackrel{\mathrm{d}}{=} B_1(X + C_1), \tag{3.1}$$

There is yet no general method to find X given arbitrary B, C, but a number of explicit cases are known. The example given here is, to the author's knowledge, a new explicit solution of (3.1).

Consider the case where $W_1 \sim \mathbf{Gamma}(3, \lambda)$ for some $\lambda > 0$ and $C_1 \sim \mathbf{Gamma}(1, 1)$. (*N.B.* The waiting times have a distribution which is the same as that of the sum of three independendent $\mathbf{Exp}(\lambda)$ variables, and we may equivalently imagine that claims arrive according to a Poisson process with parameter λ, and that only claims number 3, 6, and so on, are discounted.) Eq.(3.1) then holds. From results Section 5 of Vervaat (1979), X has finite moments of all order, and

$$x_n = \mathsf{E}X^n = b_n\left(\sum_{j=0}^{n}\binom{n}{j}x_j(n-j)!\right), \qquad b_n = \mathsf{E}B^n, \qquad n \geq 0. \tag{3.2}$$

This recursive equation uniquely determines the moments of X, because $0 < b_n < 1$ for all $n \geq 1$. Define

$$\beta = \lambda/r, \qquad \gamma = 1 + \frac{\beta}{2}(3 + i\sqrt{3})$$

and consider the sequence

$$\xi_n = \frac{(\beta)_n^3}{(\gamma)_n(\overline{\gamma})_n}.$$

We first show that this sequence satisfies (3.2). Noting that

$$b_n = \frac{\beta^3}{(\beta+n)^3},$$

we need to show that

$$\xi_n = n!\frac{\beta^3}{(\beta+n)^3}\sum_{j=0}^{n}\frac{\xi_j}{j!},$$

which is equivalent to

$$\frac{(\beta)_n^3}{(\gamma)_n(\overline{\gamma})_n}\frac{(\beta+n)^3}{n!\beta^3} \;=\; \sum_{j=0}^{n}\frac{1}{j!}\frac{(\beta)_j^3}{(\gamma)_j(\overline{\gamma})_j}.$$

Proceed by induction. The result is correct for $n=0$; suppose it holds for $n=0,1,\ldots,N$. It remains to verify that

$$\frac{(\beta)_N^3}{(\gamma)_N(\overline{\gamma})_N}\frac{(\beta+N)^3}{N!\beta^3}+\frac{1}{(N+1)!}\frac{(\beta)_{N+1}^3}{(\gamma)_{N+1}(\overline{\gamma})_{N+1}} \;=\; \frac{(\beta)_{N+1}^3}{(\gamma)_{N+1}(\overline{\gamma})_{N+1}}\frac{(\beta+N+1)^3}{(N+1)!\beta^3}.$$

Cancelling common factors, this identity is the same as

$$(N+1)(\gamma+N)(\overline{\gamma}+N)+\beta^3 \;=\; (\beta+N+1)^3,$$

which is easily checked. We have thus proved that the sequence $\{\xi_n\}$ satisfies (3.2).

Now, ξ_n is the n-th moment of $Y=GH$, where $G\sim \mathbf{BetaP}(\beta,\gamma-\beta)$ and $H\sim \mathbf{Gamma}(\beta)$ are independent. The distribution of Y is determined by its moments, since

$$\mathsf{E}e^{sY} \;=\; \sum_{n=0}^{\infty}\frac{(\beta)_n^3}{(\gamma)_n(\overline{\gamma})_n}\frac{s^n}{n!} \;=\; {}_3F_2(\beta,\beta,\beta;\gamma,\overline{\gamma};s)$$

for $s<1$. The identity in law (3.1) has a unique solution X, which has the same moments as Y. The conclusion is that $X\sim \mathbf{BetaP}(\beta,\gamma-\beta,\beta,\gamma-\beta)\odot\mathbf{Gamma}(\beta)$. Finally, the density of X is

$$f_X(x) \;=\; \int_0^1 \frac{du}{u}g_{\beta,\gamma-\beta,\beta,\gamma-\beta}(u)\left(\frac{x}{u}\right)^{\beta-1}e^{-x/u}\,\mathbf{1}_{\{x>0\}}.$$

This example bears some similarities with the model studied in Chamayou & Dunau (2002). Other results on the identity $X\overset{\mathrm{d}}{=}B(X+C)$ will be given in Dufresne (2007), as well as more details on how to solve the recursive equation for the moments $\{x_n\}$.

4. Complex gamma convolutions

Since beta and gamma distributions are intimatley related, a natural question to ask next is whether the same would happen for gamma densities. Consider the functions

$$h_a(x) \;=\; \frac{x^{a-1}}{\Gamma(a)}e^{-x}\mathbf{1}_{\{x>0\}}, \qquad \mathsf{Re}(a)>0.$$

If $\mathrm{Im}(a)\neq 0$, then this is not a probability density. Nonetheless, the Mellin transform of the multiplicative convolution of h_a and $h_{\overline{a}}$ is real for real arguments r, and equals

$$\int dy\, y^r(h_a\odot h_{\overline{a}})(y) \;=\; \left|\frac{\Gamma(a+r)}{\Gamma(a)}\right|^2, \qquad r\in\mathbb{R}.$$

Therefore, $h_a \odot h_{\overline{a}}$ is real. This function may be expressed in terms of Bessel functions: using the formulas

$$K_\nu(z) \;=\; \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi} \;=\; \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t-z^2/4t} t^{-\nu-1}\, dt,$$

which relate the modified Bessel functions of the first and third kind (Lebedev, 1972, p.119; the second identity holds whenever $\mathsf{Re}(z^2) > 0$), we get (letting $\beta = 2\mathrm{Im}(a)$)

$$\begin{aligned}(h_a \odot h_{\overline{a}})(y) &= \int_0^\infty \frac{dx}{x} h_a(x) h_{\overline{a}}\Big(\frac{y}{x}\Big)\\ &= \frac{y^{\overline{a}-1}}{|\Gamma(a)|^2}\int_0^\infty dx\, x^{i\beta-1}e^{-x-y/x}\\ &= \frac{y^{\overline{a}-1}}{|\Gamma(a)|^2} 2(\sqrt{y})^{i\beta} K_{-i\beta}(2\sqrt{y})\\ &= -\frac{2\pi y^{\mathsf{Re}(a)-1}}{|\Gamma(a)|^2 \sinh(\beta\pi)} \mathrm{Im}(I_{i\beta}(2\sqrt{y})).\end{aligned}$$

This is a real number for any $y > 0$, and, from

$$I_\nu(z) \;=\; \sum_{k=0}^\infty \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}$$

we find that

$$(h_a \odot h_{\overline{a}})(y) \;\sim\; -\frac{2\pi y^{\mathsf{Re}(a)-1}}{|\Gamma(a)|^2 \sinh(\beta\pi)} \mathrm{Im}\left(\frac{y^{i\beta/2}}{\Gamma(1+i\beta)}\right)$$

as $y \downarrow 0$. The last expression changes sign endlessly as y tends to 0, and thus *the convolution of h_a and $h_{\overline{a}}$ is not a probability density if* $\mathrm{Im}(a) \neq 0$.

There is an interesting curiosity related to the above negative result. Suppose once again that $\mathsf{Re}(a) > 0$, $\mathrm{Im}(a) \neq 0$. Since the Mellin transform of the multiplicative convolution of $h_a \odot h_{\overline{a}}$ with the $\mathbf{BetaP}(b, a-b, b, \overline{a}-b)$ density, $0 < b < \mathsf{Re}(a)$, is

$$\left|\frac{\Gamma(a+r)}{\Gamma(a)}\right|^2 \left|\frac{\Gamma(b+r)\Gamma(a)}{\Gamma(b)\Gamma(a+r)}\right|^2 \;=\; \left|\frac{\Gamma(b+r)}{\Gamma(b)}\right|^2.$$

The right-hand side is the Mellin transform of $h_b \odot h_b$, which is of course a true probability density. We conclude that the last equation is an instance of the convolution of a probability density with a function which is not a probability density, the outcome of which is a true probability density:

$$h_a \odot h_{\overline{a}} \odot g_{b,a-b,b,\overline{a}-b} \;=\; h_b \odot h_b.$$

When $a \in \mathbb{R}_{++}$, this of course reduces to the well-known identity in law

$$\mathbf{Gamma}(a) \odot \mathbf{Beta}(b, a-b) \;=\; \mathbf{Gamma}(b).$$

The same occurs with beta product distributions: the first function in

$$g_{a+b,c,a+\overline{b},\overline{c}} \odot g_{a,b,a,\overline{b}} \;=\; g_{a,b+c,a,\overline{b}+\overline{c}}$$

is not a probability density, but the other two are, if $a, \text{Re}(b), \mathsf{Re}(c) > 0$, $\text{Im}(b) \neq 0$.

5. Complex beta and gamma convolutions

Suppose $\mathsf{Re}(a), \mathsf{Re}(b), \mathsf{Re}(c) > 0$ and consider the multiplicative convolution of the functions $f_{a,b}$ and h_c. The Mellin transform of the outcome is

$$m(r) = \frac{\Gamma(a+r)\Gamma(c+r)\Gamma(a+b)}{\Gamma(a+b+r)\Gamma(a)\Gamma(c)}, \tag{5.1}$$

and the ratio

$$k(r) = \frac{m(r+1)}{m(r)} = \frac{(a+r)(c+r)}{a+b+r}.$$

is real at least for all $r \geq 0$. Multiplying $k(r)$ by $a+b+r$ and differentiating twice tells us that $a+b$ is real, which in turn implies that there are two cases. Considering the first derivative of $(a+b+r)k(r)$ then says that $a+c$ is real. This in turn implies that $ac \in \mathbb{R}$ and so $a = \overline{c}$. If $\text{Im}(a) = 0$ then we conclude that $a, b, c > 0$. If $\text{Im}(a) \neq 0$, then (5.1) says that $\mathsf{E}X^r$ is finite for all $r \in \mathbb{R}$, and in particular $\mathsf{E}X^{-a-b} = 0$, which is impossible. It was then incorrect to assume $\text{Im}(a) \neq 0$.

Theorem 3. *Consider the G distributions on* $(0, \infty)$ *denoted*

$$\begin{pmatrix} a & b \\ & - \end{pmatrix}, \qquad \begin{pmatrix} a & c \\ a+b & \end{pmatrix},$$

where a, b, c are allowed to take arbitrary complex values, and $a, c \neq a+b$. Then a, b, c must be real and positive, and the two G distributions are the same as, respectively, **Gamma**$(a) \odot$ **Gamma**(b) *and* **Beta**$(a, b) \odot$ **Gamma**(c).

Proof. In the first case, it is easy to see that the following must hold: $\mathsf{Re}(a), \mathsf{Re}(b) > 0$; either (1) $a, b > 0$ or (2) $a = \overline{b}$. The rest was spelled out in Section 4. In the second case it is clear that we must have $\mathsf{Re}(a), \mathsf{Re}(c) > 0$, and the arguments given above finish the proof. □

References

Bhargava, R.P., and Khatri, C.G. (1981). The distribution of product of independent beta random variables with applications to multivariate analysis. *Annals of the Institute of Statistical Mathematics* **33**: 287-296.

Chamayou, J.-F., and Dunau, J.-L. (2002). Random difference equations with logarithmic distribution and the triggered shot noise. *Advances in Applied Math.*: **29** 454-470.

Chamayou, J.-F., and Letac, G. (1991). Explicit stationary distributions for compositions of random functions and product of random matrices. *J. Theor. Probab.* **4**: 336.

Dufresne, D. (1996). On the stochastic equation $\mathcal{L}(X) = \mathcal{L}[B(X+C)]$ and a property of gamma distributions. *Bernoulli* **2**: 287-291.

Dufresne, D. (1998). Algebraic properties of beta and gamma distributions, and applications. *Advances in Applied Mathematics* 20: 285-299.

Dufresne, D. (2007). Rational distributions and explicit solutions of $X =$(in law)$B(X+C)$. In preparation.

Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F.G. (1953). *Higher Transcendental Functions. Vol. I.* McGraw-Hill, New York, 1953.

Fan, D.Y. (1991). The distribution of the product of independent beta random variables. *Communications in Statistics – Theory and Methods* **20**: 4043-4052.

Jambunathan, M.V. (1954). Some properties of beta and gamma distributions. *Annals of Mathematical Statistics* **25**: 401-405.

Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acat Math.* **131**: 207-248.

Lebedev, N.N. (1972). *Special Functions and their Applications.* Dover, New York.

Nadarajah, S. (2005). Sums, products, and ratios of non-central beta variables. *Communications in Statistics – Theory and Methods* **34**: 89 - 100.

Pham-Gia, T. (2000). Distributions of the ratios of independent beta variables and applications. *Communications in Statistics – Theory and Methods* **29**: 2693-2715.

Springer, M.D., and Thompson, W.E. (1970). The distribution of products of beta, gamma and Gaussian random variables. *SIAM J. Appl. Math.* **18**: 721-737.

Tang, J., and Gupta, A.K. (1984). Distribution of the product of independent beta random variables. *Statistics and Probability Letters* **2**: 165-168.

Vervaat, W. (1979). On a stochastic difference equation and a representation of non negative infinitely divisible random variables *Adv. Appl. Probab.* **11**: 750-783.

Widder, D.V. (1941). *The Laplace Transform.* Princeton University Press.

Appendix: Inequality (2.5)

The inequality used here is: for fixed $\alpha,\beta,\gamma\in\mathbb{R}$, there is $n_0<\infty$ such that $|(\alpha+n+i\gamma)(\beta+n-i\gamma)|\le|z|^2$ for all $n\ge n_0$, if $z=(\alpha+\beta)/2+n+i\gamma$. We need to show that for n large enough we have $L_n\le R_n$, with

$$L_n \;=\; |(\alpha+n+i\gamma)(\beta+n-i\gamma)|^2, \qquad R_n \;=\; \left|\left(\frac{\alpha+\beta}{2}\right)+n+i\gamma\right|^4.$$

Expanding, we find

$$\begin{aligned} L_n &= [(\alpha+n)^2+\gamma^2][(\beta+n)^2+\gamma^2] \;=\; (\alpha+n)^2(\beta+n)^2+\gamma^2[(\alpha+n)^2+(\beta+n)^2]+\gamma^4 \\ R_n &= \left(\frac{\alpha+\beta}{2}+n\right)^4+2\gamma^2\left(\frac{\alpha+\beta}{2}+n\right)^2+\gamma^4. \end{aligned}$$

Then:

$$\left(\frac{\alpha+\beta}{2}+n\right)^4-(\alpha+n)^2(\beta+n)^2$$
$$= \left[\left(\frac{\alpha+\beta}{2}+n\right)^2-(\alpha+n)(\beta+n)\right]\left[\left(\frac{\alpha+\beta}{2}+n\right)^2+(\alpha+n)(\beta+n)\right]$$
$$= \frac{(\alpha-\beta)^2}{4}\left[2n^2+2n(\alpha+\beta)+\frac{(\alpha+\beta)^2}{4}+\alpha\beta\right]$$

$$2\gamma^2\left(\frac{\alpha+\beta}{2}+n\right)^2-\gamma^2[(\alpha+n)^2+(\beta+n)^2] \;=\; -\frac{\gamma^2}{2}(\alpha-\beta)^2.$$

Hence, $R_n - L_n \;\sim\; (\alpha-\beta)^2 n^2/2$ as $n\to\infty$, which implies that for $n > n_0$

$$|(b)_n(d)_n| \;\le\; \left|\frac{(b)_{n_0}(d)_{n_0}}{(b')_{n_0}(d')_{n_0}}\right| |(b')_n(d')_n| \;=\; K|(b')_n(d')_n|.$$

Then

$$|{}_2F_1(b,d;b+d;1-u)| \;\le\; \sum_{n=0}^{n_0}\frac{|(b)_n(d)_n|}{(b+d)_n}\frac{(1-u)^n}{n!}+\sum_{n=n_0}^{\infty}\frac{(b')_n(d')_n}{(b+d)_n}\frac{(1-u)^n}{n!}$$
$$\le\; K'+K\,{}_2F_1(b',d';b+d;1-u)$$

where K' is an upper bound for the polynomial of degree n_0 obtained, for $0\le u\le\epsilon$.

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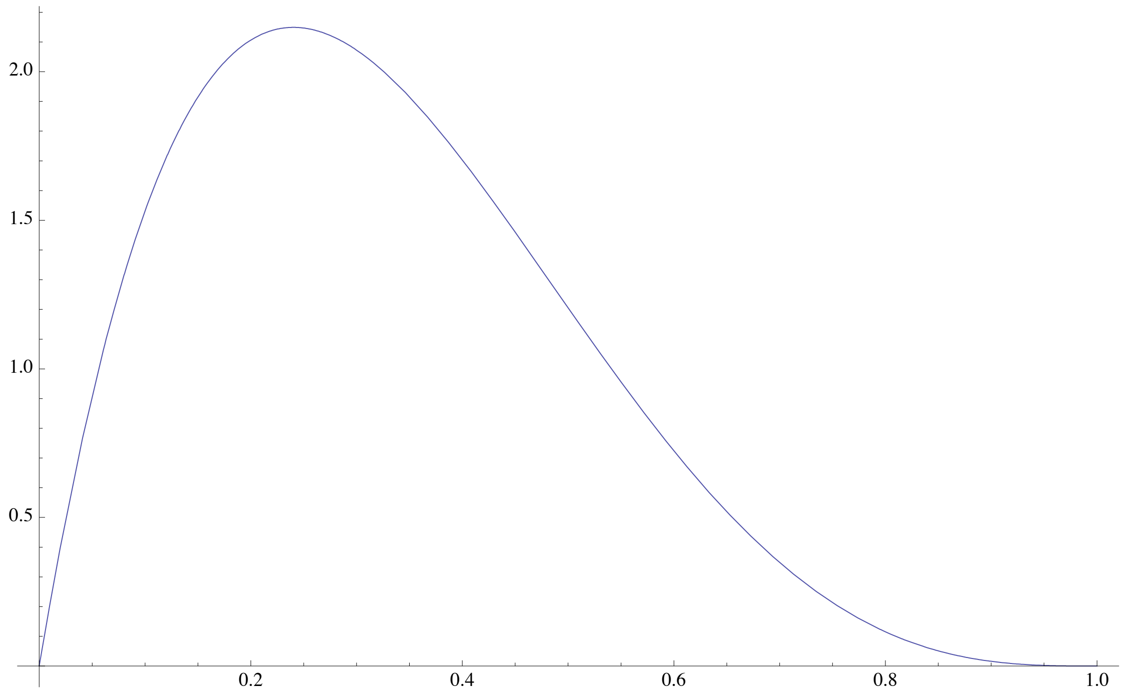

Figure 1. The BetaP$(\mathbf{2},\mathbf{5},\mathbf{8},-\mathbf{1})$ density function.

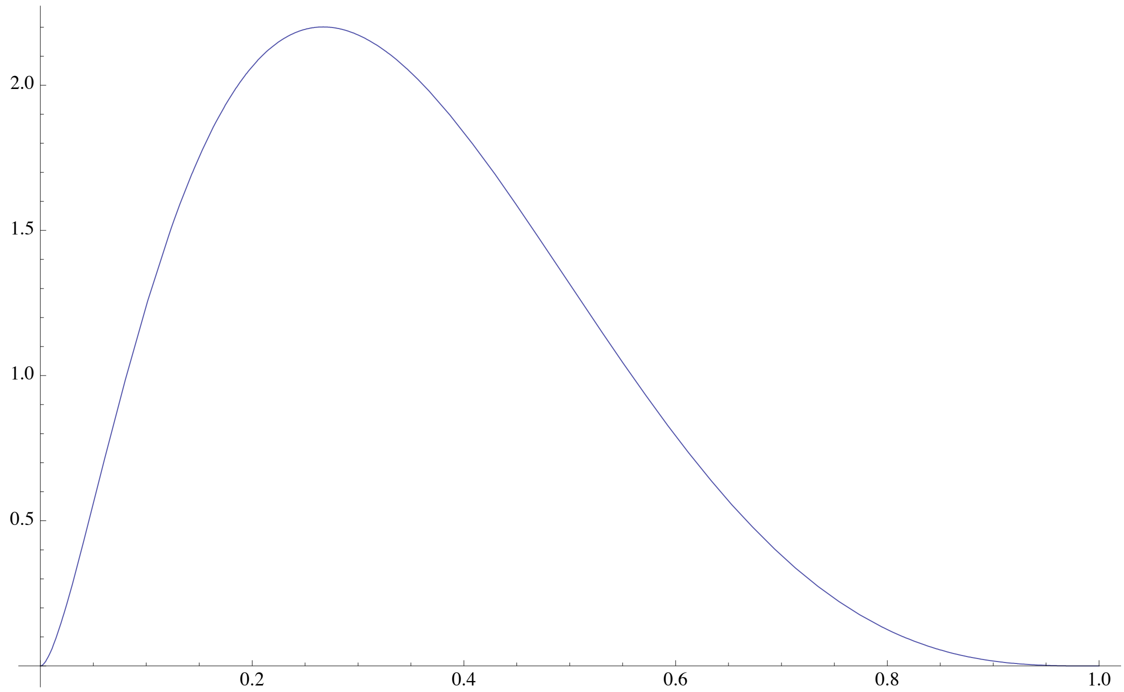

Figure 2. The function $\mathbf{g_{3+i/10,2+i,3-i/10,2-i}(u)}$, $\mathbf{0 < u < 1}$.

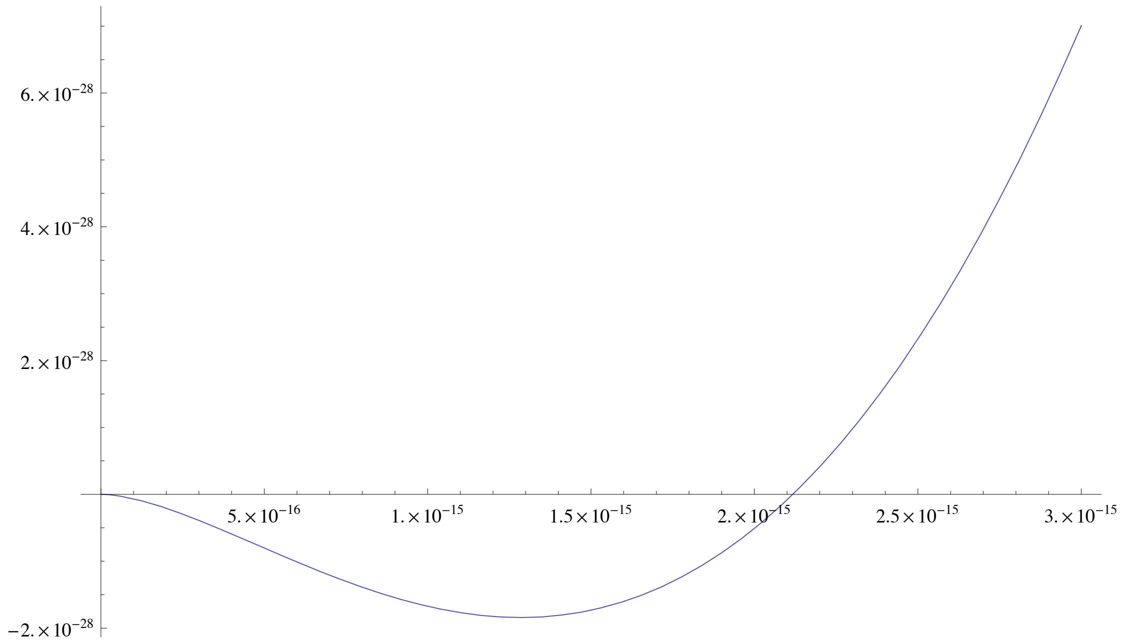

Figure 3. **The function $g_{3+i/10,2+i,3-i/10,2-i}(u)$, $0 < u < 3 \times 10^{-15}$.**