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FROM COMPOUND INTEREST TO ASIAN OPTIONS

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Abstract

This paper describes the relationship between two problems, one actuarial, the other financial: (1) the distribution of discounted values when rates of interest are random, and (2) the pricing of Asian (or average) options. It will be seen that the two problems are very similar, and that some of the results published in actuarial journals on stochastic discounting are directly relevant to the evaluation of Asian options. The concepts of random walk, Brownian motion and stochastic integration are briefly described.

Keywords: Actuarial Mathematics, Option Pricing, Stochastic Discounting

1. Introduction

The first part of the paper (Sections 2 to 5) deals with the discrete-time case: stochastic discounting, random walks, Asian options, duality between discounting and accumulating. The second part of the paper (Sections 6 to 11) is concerned with the continuous-time case: Brownian motion, stochastic integration, stochastic differentials and Itô's Formula, stochastic differential equations, stock prices and options, stochastic discounting and accumulating. Finally, Section 12 summarizes what is known on the numerical evaluation of Asian options.

2. Stochastic Discounting

Actuarial Science tells us that, in the valuation of premiums and reserves, we should not use a single estimate (for example life expectancy) for the duration of life. The correct approach is to view time of death as a random variable, the latter taking many different values with certain probabilities. It is only natural that actuaries, the traditional experts in all quesions related to compound interest, also thought of turning the *rate of interest* into a random variable.

In 1971, John Pollard published the first paper on stochastic discounting, "On fluctuating interest rates" (reference [9]). Many others have since been published in actuarial journals.

Suppose G_n is the *geometric rate of interest* (also called *force of interest*) for the period $(n-1, n)$. The discounted value of one unit at time n is then

$$e^{-G_1} \times \cdots \times e^{-G_n} \;=\; e^{-(G_1+\cdots+G_n)}.$$

We assume that the $\{G_n\}$ are random variables, so the discounted value above is also random. The value of an N-year annuity certain (with payments at the end of the year) is thus

$$a(N; G_1, \ldots, G_N) \;=\; \sum_{n=1}^{N} e^{-(G_1+\cdots+G_n)}. \tag{1}$$

(We will shorten this expression to $a(N)$ when convenient.) The exact distribution of the random variable $a(N)$ is not known. It is often possible to obtain some (or all) of its moments, however, and this was the subject of a number of actuarial papers. For example, Pollard and others made the assumption that G follows an autoregressive process:

$$G_n - \mu \;=\; \alpha_1(G_{n-1}-\mu) + \alpha_2(G_{n-2}-\mu) + \varepsilon_n,$$

where $\{\varepsilon_n\}$ are independent normal random variables, and μ is the long-term average rate of interest. This assumption makes the geometric rates of interest $\{G_n\}$

stochastically *dependent*, since the value of G_n obviously influences G_{n+1}, G_{n+2}, and so on. In this particular case it is possible (with some effort) to obtain the mean and variance of $a(N)$ explicitly in terms of the parameters of the model; higher moments may also be calculated, but the expressions become more complicated.

Remark. The moments of a random variable X are $\mathsf{E}X^n$, $n = 1, 2 \ldots$ The more moments of a random variable we know, the better we should be able to approximate its distribution. Nevertheless, knowledge of all the moments of a random variable does not in general imply exact knowledge of its distribution. □

Observing that the autoregressive assumption, though consistent with observed long-term interest rates, made calculations very difficult, some actuarial researchers made the more restrictive assumption that the $\{G_n\}$ were *independent*. This allowed the calculation of all the moments of $a(N)$. There is still hope that the distribution of $a(N)$ may be found explicitly in some cases, but this has not been achieved yet. In Section 5 it will be shown how to compute the distribution of $a(N)$ recursively.

3. A Random Walk Model

In the Theory of Probability, *a random walk is the sequence of cumulative sums of independent random variables*. For instance, suppose the variables $\{U_n\}$ are independent. Then

$$X_n \;=\; U_1 + \cdots + U_n, \quad n = 1, 2, \ldots$$

constitutes a random walk.

Remark. In the financial literature there is no consensus regarding the definition of a random walk. In some papers the meaning of the expression is the same as above, in others it may be quite different; to some authors a random walk apparently denotes any stochastic process, whether discrete-time or continuous-time. □

In Finance, a very common hypothesis is that stock prices $\{P_n\}$ behave as the exponential of a random walk:

$$P_n \;=\; P_0 e^{U_1 + \cdots + U_n} \;=\; P_{n-1} \times e^{U_n}. \tag{2}$$

This assumption is extremely convenient for calculations. It implies that stock prices never become negative, which is natural. This *multiplicative random walk hypothesis* is supported by empirical evidence, at least in the case of stocks. It does not represent all asset prices adequately; for instance, observed bond prices do not agree with this model.

There is an obvious similarity between the stock price P_n and the discount factor $e^{-(G_1+\cdots+G_n)}$ of the previous section. In fact, if the $\{G_n\}$ are independent, then discount factors also follow a multiplicative random walk.

4. Asian Options

A *European* call option on an asset with price process P, maturity N and exercise price K is worth

$$(Y-K)_+ \;=\; \begin{cases} Y-K & \text{if } \; Y > K \\ 0 & \text{if } \; Y \le K \end{cases}$$

at maturity, where Y is some specified combination of the prices P_n for $n \le N$. (We will not be concerned with *American* options, which may be exercised at any time before maturity.) A put option is worth $(K-Y)_+$ at maturity.

An *Asian* or *average* option has a payoff related to the average value of the underlying asset over some specified period. (Asian options may be European or American — Finance has reinvented geography. We will concentrate on European-style Asian options.) For example, an Asian call option may specify a payoff equal to the average price over the period $[0,N]$ minus a pre-determined exercise price K, if the result is positive:

$$\text{value at maturity (time } N) \;=\; \begin{cases} \frac{1}{N}\sum_{n=1}^{N} P_n - K & \text{if this is positive} \\ 0 & \text{otherwise.} \end{cases}$$

According to Geman & Yor ([5], p.353) :

> [...] Asian options are popular in the financial community, because they might be superior to standard options for several reasons. First, [...] they are often cheaper than the equivalent classical European option (or are thought to be so even when it is not the case). Second, the fact that the option is based on an average price is an attractive feature for thinly traded assets and commodities (e.g. gold and crude oil) where price manipulations near the option expiration date are possible [...]. Today, average options represent an enormous percentage of those on oil; some are written directly on oil prices, others on spreads between two types of oil. In the same way, Asian options on average exchange rates are extremely popular because they are less expensive for corporations which need to hedge a series of risky positions on a foreign exchange incurring at a steady rate over a period of time.

The value of an Asian call option is given by

$$e^{-rN} \mathsf{E} \left(\frac{1}{N} \sum_{n=1}^{N} P_n - K \right)_+ \tag{3}$$

where r is the risk-free rate or interest and the expectation is computed according to the so-called "risk neutral measure" (see for example Hull [7]).

To compute this expected value, it is necessary to know the distribution of the sum of the prices $\mathcal{A}_N = \sum_{n=1}^{N} P_n$. Now we have seen that the prices $\{P_n\}$ are modelled as a multiplicative random walk. Thus, *finding the values of (European-style) Asian options is the same problem as finding the distribution of the stochastically discounted value* $\{a(N)\}$, *when the discount factors are assumed independent.*

5. Duality Between Discounting and Accumulating

The discounted value in Eq. (1) satisfies

$$a(N; G_1, \ldots, G_N) \;=\; e^{-G_1}[1 + a(N-1; G_2, \ldots, G_N)]. \tag{4}$$

In the case where the $\{G_n\}$ are independent and identically distributed (i.i.d.), this yields a recursive equation for the distribution of $a(N)$. Changing the order of

$G_1, \ldots, G_N$ on the right-hand side of (4) does not change the resulting distribution; in particular,

$$\mathrm{Dist}\{a(N; G_1, \ldots, G_N)\} \;=\; \mathrm{Dist}\{e^{-G_N}[1 + a(N-1; G_1, \ldots, G_{N-1})]\}.$$

But e^{-G_N} is independent of $a(N-1; G_1, \ldots, G_{N-1})$. Hence the distribution of $a(N)$ is obtained by translating the distribution of $a(N-1)$ by one unit, and then performing the product convolution of the result with the distribution of e^{-G_N}. For instance, the moments of the discounted values satisfy

$$\begin{aligned} \mathsf{E}a(N)^k &= \mathsf{E}e^{-kG_N}\mathsf{E}[1 + a(N-1)]^k \\ &= \mathsf{E}e^{-kG_N}\left[1 + \sum_{j=1}^{k}\binom{k}{j}\mathsf{E}a(N-1)^j\right] \end{aligned}$$

for $k = 1, 2 \ldots$

The simulation of $a(N)$ (in the i.i.d. case) may therefore be done in two ways. The first one uses the recursion

$$a(n; G_1, \ldots, G_n) \;=\; a(n-1; G_1, \ldots, G_{n-1}) + e^{-(G_1 + \cdots + G_n)}.$$

The program needs to carry the vector $(a(n), e^{-(G_1+\cdots+G_n)})$ for $n = 1, 2, \ldots$ The other one involves defining a new variable

$$b(n; G_1, \ldots, G_n) \;=\; e^{-G_n} + \cdots + e^{-(G_1+\cdots+G_n)} \;=\; a(n; G_n, \ldots, G_1).$$

Then $a(n)$ and $b(n)$ have the same distribution for each fixed n, and the latter may be simulated using the recursion

$$b(n) \;=\; e^{-G_N}[1 + b(n-1)]. \tag{5}$$

What precedes may be given a more general interpretation. Recall the usual definitions

$$\begin{aligned} a_{\overline{n}|\delta} &= e^{-\delta} + \cdots + e^{-n\delta} &&= \frac{1 - e^{-n\delta}}{e^{\delta} - 1} \\ \ddot{a}_{\overline{n}|\delta} &= 1 + \cdots + e^{-(n-1)\delta} &&= \frac{1 - e^{-n\delta}}{1 - e^{-\delta}} \\ s_{\overline{n}|\delta} &= 1 + \cdots + e^{(n-1)\delta} &&= \frac{e^{n\delta} - 1}{e^{\delta} - 1} \\ \ddot{s}_{\overline{n}|\delta} &= e^{\delta} + \cdots + e^{n\delta} &&= \frac{e^{n\delta} - 1}{1 - e^{-\delta}}. \end{aligned}$$

There are recursive expressions for those present values. An example is

$$a_{\overline{n}|\delta} \;=\; a_{\overline{n-1}|\delta} + e^{-n\delta}. \tag{6}$$

In words, the discounted value of an annuity-immediate for n years is the value of the same annuity for $n-1$ years, plus the discounted value of the additional nth unit. Similarly, the accumulated value of an annuity-due for $n-1$ years, plus the nth unit, increased with interest, yields the accumulated value of an annuity for n years. In symbols:

$$\ddot{s}_{\overline{n}|\delta} = (\ddot{s}_{\overline{n-1}|\delta} + 1)e^{\delta}. \tag{7}$$

Formulas (6) and (7) are different, but a little thought shows that the same principles may be applied to either discounted or accumulated values:

$$\ddot{s}_{\overline{n}|\delta} \;=\; \ddot{s}_{\overline{n-1}|\delta} + e^{n\delta}. \tag{8}$$

(counterpart of (6) for accumulation) and

$$a_{\overline{n}|\delta} \;=\; (a_{\overline{n-1}|\delta} + 1)e^{-\delta} \tag{9}$$

(counterpart of (7) for discounting). Eqs. (6) to (9) have intuitive interpretations, which have to do with moving "forward" or "backward" in time. The same types of formulas are obtained for discounted or accumulated values, because discounting and accumulating at a constant rate of interest are essentially the same thing:

$$a_{\overline{n}|\delta} \;=\; \ddot{s}_{\overline{n}|-\delta} \tag{10}$$

$$\ddot{a}_{\overline{n}|\delta} \;=\; s_{\overline{n}|-\delta}. \tag{11}$$

This *duality* between accumulating and discounting subsists when rates of interest are variable, with the difference that the order of the rates of interest must now be reversed. Indeed, define random stochastic counterparts of $a_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $s_{\overline{n}|}$, and $\ddot{s}_{\overline{n}|}$ as in Eq. (1) and the following:

$$\begin{aligned}
\ddot{a}(n; G_1, \ldots, G_{n-1}) &= 1 + e^{-G_1} + \cdots + e^{-(G_1+\cdots+G_{n-1})} \\
s(n; G_2, \ldots, G_n) &= 1 + e^{G_n} + \cdots + e^{G_n+\cdots+G_2} \\
\ddot{s}(n; G_1, \ldots, G_n) &= e^{G_n} + \cdots + e^{G_n+\cdots+G_1}.
\end{aligned}$$

Then the counterparts of (10) and (11) are

$$a(n; G_1, \ldots, G_n) = \ddot{s}(n; -G_n, \ldots, -G_1) \tag{12}$$

$$\ddot{a}(n; G_1, \ldots, G_{n-1}) = s(n; -G_{n-1}, \ldots, -G_1). \tag{13}$$

Observe that the counterpart of (7) is

$$\ddot{s}(n; G_1, \ldots, G_n) = [\ddot{s}(n-1; G_1, \ldots, G_{n-1}) + 1]e^{G_n}. \tag{14}$$

Eqs. (12) and (14) yield Eq. (5).

6. Brownian Motion

We now turn to the continuous-time case. The following definition is fundamental in modern Finance.

Definition. *The process $\{W_t\}$ is a* ***standard Brownian motion (SBM)*** *if*

(1) $W_0 = 0$. (starts from 0)

(2) $W_{t_1} - W_{t_0} \sim \mathbf{N}(0, t_1 - t_0), \quad 0 \le t_0 < t_1$. (normality)

(3) $\{W_{t_{k+1}} - W_{t_k}; k = 0, \ldots, n-1\}$ *are independent.* (independent increments)

(4) W_t *is continuous in* t. (continuity of paths)

A common transformation of standard Brownian motion is of the form $X_t = \mu t + \sigma W_t$. The mean of X_t is μt, and its variance $\sigma^2 t$. The process X is simply called *Brownian motion*; μ is called the *drift* of X, and σ^2 its *infinitesimal variance.*

BM is a continuous-time equivalent of a random walk. It has some very interesting properties; for instance,

- X is not differentiable anywhere;
- each path of $\{X_t; 0 \le t \le T\}$ has infinite length.

7. Stochastic Integration

Before introducing stochastic differentials, it is necessary to define stochastic integration. The symbol $\mathcal{F}_t$ represents the information given by $\{W_t; 0 \le s \le t\}$. The integral of a process Y with respect to a standard Brownian motion W, written $\int_0^T Y_t\, dW_t$, is only defined for a certain class of processes Y. One restriction is that Y_t "does not depend on the future", which mathematicians express as "$\{Y_t\}$ is adapted to $\{\mathcal{F}_t\}$". (In financial models this condition is usually satisfied.) Although this is not required in the general theory, we simplify our brief description by making the additional assumption that the paths of Y are continuous.

Examples of adapted processes with continuous paths include $Y_t^{(1)} = W_t$ and $Y_t^{(2)} = \int_0^t W_s\, ds$.

The stochastic integral is the limit of a sum. Consider partitions

$$\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$$

of the interval $[0, T]$ and define

$$\mathcal{S}^\pi(Y) \;=\; \sum_{k=0}^{n-1} Y_{t_k}(W_{t_{k+1}} - W_{t_k}), \qquad \|\pi\| \;=\; \max_k (t_{k+1} - t_k).$$

Remark. In the sum above, Y has to be evaluated at point t_k, otherwise a different integral is obtained. For instance, using $\frac{1}{2}(t_k + t_{k+1})$ instead of t_k yields the so-called *Stratonovich stochastic integral*, which does not obey Itô's Formula.

Definition. $$\int_0^T Y_t\, dW_t \;=\; \lim_{\|\pi\|\to 0} \mathcal{S}^\pi(Y) \;=\; \lim_{\|\pi\|\to 0} \sum_{k=0}^{n-1} Y_{t_k}(W_{t_{k+1}} - W_{t_k}).$$
The limit is in the sense of convergence in probability.

The relationship between stochastic integration and the theory of martingales runs very deep. Very loosely speaking, a martingale is a "process with no conditional drift".

Definition. *A process $\{M_t\}$ is a* ***martingale*** *if*

(1) *$\{M_t\}$ is adapted to $\{\mathcal{F}_t\}$,*

(2) *$\mathsf{E}|M_t| < \infty$ for all t,*

(3) *$\mathsf{E}(M_t \,|\, \mathcal{F}_s) \;=\; M_s$ for all $s < t$.*

The last requirement, that the conditional expectation of M_t given what is known at time s be equal to M_s, says that the best prediction we can give for M at some subsequent time is its current value.

Theorem. *If $\mathsf{E}\int_0^T Y_t^2\,dt < \infty$ then $M_t = \int_0^t Y_s dW_s$ is a martingale.*

8. Stochastic Differentials and Itô's Formula

Now that stochastic integrals have been defined, it is possible to consider processes which satisfy identities such as

$$X_t - X_0 \;=\; \int_0^t U_s ds + \int_0^t V_s dW_s.$$

(They are called *Itô processes*). A short-hand way to write the above is

$$dX_t \;=\; U_t\,dt + V_t\,dW_t.$$

The first expression is the integral form, and the second one the differential form. (The latter does not mean that the derivative of the Brownian motion W is involved in any way.) The right-hand side of the last equation is called the *stochastic differential* of X.

Itô's Formula is a rule to find the differential of $f(X_t)$:

$$\begin{aligned} df(X_t) \;&=\; \underbrace{f'(X_t)\,dX_t}_{(A)} + \underbrace{\tfrac{1}{2}f''(X_t)V_t^2\,dt}_{(B)} \\ &=\; \left[f'(X_t)U_t + \tfrac{1}{2}f''(X_t)V_t^2\right]dt + f'(X_t)V_t\,dW_t. \end{aligned}$$

Part (A) of the formula is the usual chain rule for derivatives. Part (B) is an additional term resulting from the specific properties of Brownian motion.

9. Stochastic Differential Equations

A *stochastic differential equation* (SDE) is a relationship of the form

$$dX_t \;=\; \alpha(X_t)dt + \beta(X_t)dW_t.$$

Example 1. The process $X_t = e^{\mu t + \sigma W_t}$ is known as *geometric Brownian motion.* An application of Itô's Formula shows that X satisfies

$$dX_t \;=\; (\mu + \tfrac{\sigma^2}{2})X_t dt + \sigma X_t dW_t.$$

Example 2. The Vasicek model for interest rates says that the short rate r satisfies

$$dr_t \;=\; -\gamma(r_t - \overline{r})dt + \sigma dW_t.$$

This is a continuous-time counterpart of the autoregressive process of order one (this process is obtained by letting $\alpha_2 = 0$ in the last equation of Section 2).

10. Stock Prices and Options

A very common assumption is that the value of one share of stock is a geometric Brownian motion $P_t = P_0 e^{\mu t + \sigma W_t}$.

Remark. This is a continuous counterpart of the multiplicative random walk $P_n \;=\; e^{U_1 + \cdots + U_n}$, $\{U_n\}$ i.i.d., described in Section 3.

A peculiar feature of this model is that the rates of return of one share of stock do not exist in the usual sense:

$$\text{“}\frac{d}{dt}\log P_t \;=\; \mu + \sigma\frac{dW_t}{dt}.\text{”}$$

The expression $\frac{dW_t}{dt}$ (the non-existent derivative of Brownian motion!) is called *white noise*. It is intuitively useful to think of white noise as the continuous-time equivalent of an i.i.d. sequence. One must of course remember that white noise does not always obey the usual rules of calculus. (The procedure to get correct answers is to use stochastic integrals and Itô's Formula.)

Now turn to options. The value (at time 0) of a call option on one share of stock with value P, having maturity T and exercise value K, now becomes

$$e^{-rT}\mathsf{E}(P_T - K)_+.$$

If we assume that averages are based on continuously sampled values of the stock (as an approximation this would be appropriate if actual averages are based on, say, daily values over several weeks), the value of the corresponding Asian call option is

$$e^{-rT}\mathsf{E}\left(\frac{1}{T}\mathcal{A}_T - K\right)_+ \quad \text{where} \quad \mathcal{A}_T = \int_0^T P_t\,dt = P_0\int_0^T e^{\mu t+\sigma W_t}\,dt.$$

Remark. As in the discrete-time case, expected values are calculated with respect to the "risk-neutral" measure. The effect is to change the drift of the Brownian motion, which becomes r instead of μ.

11. Stochastic Discounting and Accumulating

We now give the countinous-time version of stochastic discounting in the traditional actuarial framework. Let δ_t be the instantaneous rate of discount ("force of interest"). The value of a T-year annuity-certain then becomes

$$a(T) \;=\; \int_0^T e^{-\int_0^t \delta_s\,ds}\,dt.$$

Now consider the particular case where the discount rates form a white noise plus a constant: "$\delta_t = \mu + \sigma\frac{dW_t}{dt}$". The proper way to interpret this statement is that the discount factor from time t back to time 0 is $e^{-(\mu t+\sigma W_t)}$. We then have

$$a(T) \;=\; \int_0^T e^{-(\mu t+\sigma W_t)}\,dt.$$

Once again there is a clear similarity with the random variable $\mathcal{A}_T$ involved in Asian options.

The duality between accumlating and discounting also holds in continuous-time. The difference is that we cannot "reverse the order of the white noise rates of discount", as was done in Section 5 (remember, we have to deal with integrals, not derivatives). The proper way about this problem is to use what is known to probabilists as *time reversal*: Brownian motion has the same distribution if we travel on a path starting from time T and move back to time 0:

$$\mathrm{Dist}\{W_t; 0 \le t \le T\} \;=\; \mathrm{Dist}\{W_T - W_{T-t}; 0 \le t \le T\}.$$

It is then straightforward to show that $\mathrm{Dist}\{a(T)\} \;=\; \mathrm{Dist}\{b(T)\}$ for each fixed T, where

$$b(T) \;=\; e^{-(\mu T + \sigma W_T)} \int_0^T e^{\mu t + \sigma W_t}\, dt.$$

Itô's Formula implies that

$$db(t) \;=\; [(-\mu + \tfrac{\sigma^2}{2})b(t) + 1]dt + \sigma b(t)\; dW_t$$

$$\Longrightarrow \quad \frac{d}{dt}\mathsf{E}a(t)^k \;=\; (-k\mu + \tfrac{k^2}{2}\sigma^2)\mathsf{E}a(t)^k + k\mathsf{E}a(t)^{k-1}.$$

All moments of $b(T)$ (the same as those of $a(T)$) can then be found explicitly. Furthermore, there are partial differential equations satisfied by the transition density of $b(\cdot)$. These added features of the continuous-time model (as compared to the discrete-time model) are of some use, but have not allowed us (this far) to find closed-form expressions for Asian option values (or for the distribution of $a(T)$). Consequently, we must resort to numerical approximations, which are the subject of the final section of this paper.

12. Numerical Evaluation of Asian Options

We briefly consider the methods available for finding the distribution of $\mathcal{A}_N$. The usual assumption is that stock prices $\{P_n\}$ follow an exponential Gaussian random walk, that is to say, the variables $\{U_n\}$ in (2) are i.i.d. normal random variables.

Simulation

It is quite straightforward to simulate Asian options using either one of the methods described in Section 5.

The geometric average of stock prices is defined as

$$\mathcal{H}_N \;=\; (P_1 \times \cdots \times P_N)^{1/N} \,.$$

The distribution of $\mathcal{H}_N$ is known exactly, since it is the exponential of a linear combination of normal random variables. The geometric average may then be used as a control variate (see Hull [7], p. 422). Note that a theorem in mathematical analysis states that the geometric mean of non-negative numbers is never larger than their arithmetic mean (see Devinatz [3], p. 349).

Fast Fourier Transform

Caverhill and Clewlow [1] have suggested using the time reversal argument of Section 5 (Eq. (4) or, equivalently, the process $\{b(n)\}$) to obtain the distribution of $A(n); 1 \leq n \leq T\}$ in a recursive fashion. They show how to use the Fast Fourier Transform to perform the required additive and multiplicative convolutions.

Edgeworth Approximation

Turnbull & Wakeman [10] suggest using the Edgeworth approximation to calculate Asian option values. This technique has been known for a long time, and has also been applied to actuarial problems (see Gerber [6], pp. 60-62). The original idea behind Edgeworth approximations is to express any probability density function as a series involving the normal density and its derivatives. The series does not necessarily converge, but the first few terms often give very good results. In the case of Asian options the method is modified by expressing the series in terms of the lognormal density and its derivatives.

Conditioning on the Geometric Mean

Curran [2] has devised a clever method for estimating an arithmetic average of lognormal variables by conditioning on its geometric average. The method uses the inequality $\mathcal{H}_N \leq \mathcal{A}_N$ to split the expectation (3) into two parts, one of which can be evaluated exactly. The other has to be integrated numerically.

Inverting the Laplace Transform

A very significant contribution to the valuation of Asian options is the Laplace transform calculated by Geman & Yor [5]. This result concerns the continuous-time version of the problem; let

$$C(T,K) \;=\; \mathsf{E}\left(\int_0^T e^{\mu s+\sigma B_s}\,ds - K\right)_+ .$$

Geman & Yor have obtained a closed form (but complicated) expression for the Laplace transform (in the variable T) of $C(T,K)$. Although some attemps have been made to invert the transform numerically, it appears that this may not be a practical solution.

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References

[1] Caverhill,A.P., and Clewlow, L.J. (1990). Flexible convolution. *Risk* **3**: 25-29.

[2] Curran,V. (1994). Valuing Asian options and portfolio options by conditioning on the geometric mean. *Management Science* **40**: 1705-1711.

[3] Devinatz, A. (1968). *Advanced Calculus.* Holt, Rinehart and Winston, New York.

[4] Dufresne, D. (1991). The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuarial J.* **1990**: 39-79.

[5] Geman, H. and Yor, M. (1993). Bessel processes, Asian options and perpetuities. *Mathematical Finance* **3**: 349-375.

[6] Gerber, H.U. (1979). *An Introduction to Mathematical Risk Theory.* S.S. Huebner Foundation for Insurance Education, Wharton School, Philadelphia.

[7] Hull, J.C. (1993). *Options, Futures, and Other Derivative Securities* (Second Edition). Prentice Hall, Englewood Cliffs, New Jersey.

[8] Lemieux, C. (1996). *Évaluation d'options asiatiques.* Master's thesis, Department of Mathematics, University of Montreal.

[9] Pollard, J.H. (1971). On fluctuating interest rates. *Bulletin de l'Association royale des actuaires belges* **66**: 68-94.

[10] Turnbull, S. and Wakeman, L. (1991). A quick algorithm for pricing European average options. *The Journal of Financial and Quantitative Analysis* **26**: 377-389.

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