Discounted Sums with Renewal Times

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Assuming that amounts are paid at renewal times, we study the discounted value of their sum over a finite or infinite time period. Mathematically this is more or less tractable depending on the distributions of amounts and waiting times. We focus on cases where waiting times are independent but not necessarily exponential. We derive a general recursive relationship for the Laplace transform of the discounted total amounts paid up to a finite time t. This yields explicit expressions for all the moments of discounted amounts up to time t. We also give new explicit distributions for both t finite and infinite.

Keywords: Generalized Ornstein-Uhlenbeck processes; ruin theory; beta-gamma algebra; Lévy processes; stochastic difference equations

1. Introduction

Financial models over an extended time period usually take the time value of money into account. The value now (time 0) of an amount X paid t years into the future is

$$X \cdot D_t \;=\; X \cdot \exp - \int_0^t r_s \, ds,$$

where D_t is the "discount factor". Here r_s is the spot rate of interest, or some other rate representing the rate of return achieved at time s. The rate of interest may be constant, variable over time or random.

The classical ruin model of risk theory assumes that the surplus process is

$$U_t \;=\; u_0 + \pi t - \sum_{j=1}^{N_t} C_j,$$

where premiums are received at a constant rate π per time unit and $\{C_j\}$ are i.i.d. claims, paid out of the fund at times $T_1 < T_2 < \ldots$, and N_t is the number of claims paid up to time t. The assumption is that assets earn no investment return, which is not realistic. Introducing a constant non-zero rate of return r leads to a modified model where the intial surplus, the premiums and the claims are accumulated to current time t:

$$\begin{aligned}\widetilde{U}_t \;&=\; e^{rt}u_0 + \pi \int_0^t e^{r(t-s)} \, ds - \sum_{j=1}^{N_t} e^{r(t-T_j)} C_j \\ &=\; e^{rt}\left(u_0 + \int_0^t e^{-rs} d\xi_s\right),\end{aligned}$$

where

$$\xi_t \;=\; \pi t - \sum_{j=1}^{N_t} C_j.$$

The process $\widetilde{U}$ is an example of *generalized Ornstein-Uhlenbeck* process. The latter allow replacing the accumulation factor e^{rt} with the exponential of a Lévy process (see Barndorff-Nielsen & Shephard (2001)), which is what we are going to do in this paper. The financial interpretation is that the return process is random. It is known that if R and ξ are independent Lévy processes then the distribution of

$$V_\infty \;=\; \int_0^t e^{-Rs} d\xi_s$$

yields the probability of ruin over an infinite period for the process $\widetilde{U}$ (Gjessing & Paulsen, 1997; Carmona *et al.*, 1994).

Define

$$Z_t \;=\; \sum_{j=1}^{N_t} e^{-R_{T_j}} C_j,$$

where (1) amount C_j is paid at time T_j; (2) $\{T_j\}$ is a renewal process, *i.e.* the waiting times $T_j - T_{j-1}$ are i.i.d; (3)$\{N_t\}$ is the number of claims up to time t (the counting process), and (4) $\{R_t\}$ (minus the logarithm of the discount process) is a Lévy process. It has been known for a long time that there are significant simplifications when $\{T_j\}$ is a Poisson process (see Section 4.1), but we are interested in explicit results in the general case where the waiting times $T_j - T_{j-1}$ are independent but have some other distribution. We assume that the waiting times are positive and that the arrival times $\{T_j\}$ are independent of the claim amounts $\{C_k\}$. The distribution of Z_t for $t < \infty$ has been studied by Aebi *et al.* (1994), Léveillé, & Garrido (2001a, 2001b), Léveillé *et al.* (2010), and Wang *et al.* (2016).

This paper has two parts. Part A derives new explicit examples of distributions of Z_∞; this extends previous work, related in particular the so-called "beta-gamma algebra". An essential tool for deriving the distribution of Z_∞ is the recursive identity in law

$$Z_\infty \stackrel{\text{dist}}{=} A(C + Z_\infty)$$

(explained in detail below). This identity has an elementary proof but is crucial in deriving differential equations or transforms that lead to identifying the distribution of Z_∞. By contrast there has up to now been no such identity for finite t, which would help study the distribution of Z_t. This is why Part B of the paper introduces Z_{S_λ}, the process Z_t sampled at an independent exponential time S_λ, and shows that there is the identity (Theorem 6)

$$Z_{S_\lambda} \stackrel{\text{dist}}{=} A^{(\lambda)}(C + Z_{S_\lambda}),$$

which does help study the properties of Z_t, at the cost of inverting the Laplace transform in λ. It is shown that the new identity gives the Laplace transform of all integer moments of Z_t. There are then explicit results on th moments of Z_t and how to calculate them, as well as on the differentiability of $\mathbf{E}\exp(sZ_t)$ with respect to t. Section 6 looks at specific cases where the distribution of $Z_t, t < \infty$, can be found explicitly.

Part A. Limit of Z_t as t tends to ∞

The limit of Z_t as $t \to \infty$ does not necessarily exist, but when it does it is almost always easier to find the distribution of Z_∞ than the distribution of Z_t for finite t.

Under the assumption that Z_∞ converges almost surely,

$$\begin{aligned} Z_\infty &= e^{-R_{T_1}} C_1 + e^{-R_{T_2}} C_2 + \cdots \\ &= A_1 C_1 + A_1 A_2 C_2 + \cdots , \end{aligned}$$

where the variables $A_j = \exp(-(R_{T_j} - R_{T_{j-1}}))$ are i.i.d. (and by assumption independent of the claims $\{C_k\}$). This type of infinite sum of products of random variables has some history in probability theory, here is an incomplete list of references: Takacs (1954, 1955, 1956), Kesten (1973), Vervaat (1979), Chamayou & Letac (1991, 1999), Carmona *et al.* (1994), Embrechts & Goldie (1994), Gjessing & Paulsen (1997), Chamayou & Dunau (2002, 2003), Bertoin *et al.* (2004), Bertoin & Yor (2005), Chamayou (2004, 2005), Dufresne (1990, 1996, 1998, 20010a, 2010b), Konstantinides & Mikosch (2005), Ostrovsky (2012), McKinley (2015), Hürlimann (2015).

If we factor out the A_1 on the right of the last equation we get

$$Z_\infty = A_1[C_1 + (A_2C_2 + A_2A_3C_3 + \cdots)].$$

The expression in round brackets has the same distribution as Z_∞, and is independent of both A_1 and C_1. Hence, we have the identity in distribution

$$Z_\infty \stackrel{\text{dist}}{=} A_1(C_1 + \widetilde{Z}_\infty), \tag{1}$$

where $\widetilde{Z}_\infty$ is a copy of Z_∞, and the variables on the right are independent.

Vervaat (1979) studies the general properties of (1) in great detail, let us only recall a sufficient condition for convergence: *if* $\mathbf{E}\log|A_1| < 0$ *and* $\mathbf{E}\log|C_1| < \infty$ *then* $Z_t(\omega)$ *converges almost surely to a finite number* $Z_\infty(\omega)$.

Beta-gamma algebra. We will write: "$G^{(a)}$" for any variable with a $\mathbf{Gamma}(a,1)$ distribution ($a > 0$); "$B^{(a,b)}$" for any variable with a $\mathbf{Beta}(a,b)$ distribution ($a, b > 0$).

A known instance of $Z_\infty \stackrel{\text{dist}}{=} A_1(C_1 + \widetilde{Z}_\infty)$ is the case

$$Z_\infty \stackrel{\text{dist}}{=} G^{(a)}, \quad A_j \stackrel{\text{dist}}{=} B^{(a,b)}, \quad C_j \stackrel{\text{dist}}{=} G^{(b)},$$

which may be rewritten as:

$$G^{(a)} \stackrel{\text{dist}}{=} B^{(a,b)}(G_1^{(a)} + G_2^{(b)}), \tag{2}$$

where the variables on the right are independent and $a, b > 0$.

There are several ways to prove (2), one is to calculate the moments of both sides of the identity: for $n = 0, 1, \ldots$,

$$\begin{aligned}
\mathbf{E}(G^{(a)})^n &= (a)_n \\
\mathbf{E}(B^{(a,b)})^n &= \frac{(a)_n}{(a+b)_n} \\
\mathbf{E}(G^{(a)} + G^{(b)})^n &= (a+b)_n \\
(a)_n &= \frac{\Gamma(a+n)}{\Gamma(a)}.
\end{aligned}$$

The beta and gamma distributions are both determined by their moments.

Another property of beta and gamma distributions is (Dufresne, 1998, Theorem 2(b))

$$B^{(a,b+c)}G_1^{(b)} + G_2^{(c)} \stackrel{\text{dist}}{=} B^{(b+c,a)}G^{(a+c)}, \tag{3}$$

where as before all variables are independent, and $a, b, c > 0$. For the sequel it is useful to give an interpretation of this property in terms of Gauss hypergeometric functions.

The Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ is the analytic continuation of the function

$$z \mapsto \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

which converges for $|z| < 1$ (Lebedev, 1972, p.238). The moment-generating function (MGF) of the right-hand side of (3) is found from its moments:

$$\begin{aligned}
M(s) &= \sum_{n=0}^{\infty} \mathbf{E}(B^{(b+c,a)}G^{(a+c)})^n \frac{s^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(b+c)_n (a+c)_n}{(a+b+c)_n} \frac{s^n}{n!} \\
&= {}_2F_1(b+c, a+c; a+b+c; s).
\end{aligned}$$

Using this result on the left of (3) gives

$$M(s) \;=\; (1-s)^{-c}{}_2F_1(a,b;a+b+c;s).$$

The identity between distributions in (3) may thus be seen as an instance of the identity between hypergeometric functions (Lebedev, 1972, p.248)

$${}_2F_1(b+c,a+c;a+b+c;s) \;=\; (1-s)^{-c}{}_2F_1(a,b;a+b+c;s).$$

It is interesting to note that (Dufresne, 1998) both identities (2) and (3) may be proved solely from the following well-known property of gamma distributions: if $G_1^{(a)}$ and $G_2^{(b)}$ are independent then the two variables

$$\frac{G_1^{(a)}}{G_1^{(a)}+G_2^{(b)}}, \qquad G_1^{(a)}+G_2^{(b)} \tag{4}$$

are also independent (the first variable has a $\mathbf{Beta}(a,b)$ distribution). The implication is that the above property of hypergeometric functions could be proved using the independence of the variables in (4) (and then analytic continuation to obtain the property of hypergeometric functions for all parameters).

Dufresne (1998, 2010b) explores the connection between the properties of beta and gamma variables, the known explicit examples of distributions of Z_∞ and special functions. We now prove some new explicit identities $Z_\infty \overset{\text{dist}}{=} A(C+\widetilde{Z}_\infty)$ that also involve beta and gamma distributions, and are related to special functions.

2. First Example

Assume that the density of $-\log A$ is a combination of exponentials:

$$w\alpha e^{-\alpha x}+\overline{w}\beta e^{-\beta x}, \quad x>0.$$

Here α,β are arbitrary positive numbers, $w,\overline{w}$ add up to 1 and are such that the function is non-negative (they do not both have to be positive, for instance $2e^{-x}-2e^{-2x}$, $x>0$, is a density function). The density of A is then

$$f(u) \;=\; w\alpha u^{\alpha-1}+\overline{w}\beta u^{\beta-1}, \quad 0<u<1.$$

We first obtain a differential equation for the MGF of Z, denoted $M_Z(\cdot)$ (the MGF of C is $M_C(s)$). (*N.B.* Initially one does not know whether the MFG exists for positive s so one may first perform the following calculations for $s<0$ or for s imaginary.) From Theorem 1.6 in Vervaat (1979) we know that there is a unique distribution for Z that solves $Z \overset{\text{dist}}{=} A(Z+C)$. We have

$$\begin{aligned}
M_Z(s) &= \int_0^1 f(u)M_Z(su)M_C(su)\,du \\
&= \int_0^s (s^{-\alpha}w\alpha v^{\alpha-1}+s^{-\beta}\overline{w}\beta v^{\beta-1})M_Z(v)M_C(v)\,dv \\
s^\beta M_Z(s) &= ws^{\beta-\alpha}\int_0^s \alpha v^{\alpha-1}M_Z(v)M_C(v)\,dv+\overline{w}\int_0^s \beta v^{\beta-1}M_Z(v)M_C(v)\,dv \\
(s^\beta M_Z(s))' &= (\alpha w+\beta\overline{w})s^{\beta-1}M_Z(s)M_C(s)+w(\beta-\alpha)s^{\beta-\alpha-1}\int_0^s \alpha v^{\alpha-1}M_Z(v)M_C(v)\,dv \\
[s^{\alpha-\beta+1}(s^\beta M_Z(s))']' &= (\alpha w+\beta\overline{w})(s^\alpha M_Z(s)M_C(s))'+w(\beta-\alpha)\alpha s^{\alpha-1}M_Z(s)M_C(s),
\end{aligned}$$

whence, letting $K_1=\alpha w+\beta\overline{w}$,

$$(\alpha\beta-\alpha\beta M_C-K_1M_C')M_Z+s(\alpha+\beta+1-K_1M_C)M_Z'+s^2M_Z'' \;=\; 0.$$

We are able to solve this equation when $M_C(s)=1/(1-s)$, as it then becomes

$$-[\alpha\beta(1-s)-K_1]M_Z+(1-s)[(\alpha+\beta+1)(1-s)-K_1]M_Z'+s(1-s)^2M_Z'' \;=\; 0.$$

The substitution $M_Z(s) = (1-s)g(s)$ yields

$$-(\alpha\beta+\alpha+\beta+1)g + [(\alpha+\beta+1)(1-s) - K_1 - 2s]g' + s(1-s)g'' \;=\; 0.$$

This is a Gauss hypergeometric equation, with general solution (Lebedev, 1972, p.248)

$$A_1\,{}_2F_1(\alpha{+}1,\beta{+}1;\overline{w}\alpha{+}w\beta{+}1;s){+}A_2\,s^{-\overline{w}\alpha-w\beta}\,{}_2F_1(1{+}w(\alpha{-}\beta),1{+}w(\beta{-}\alpha);1{-}\overline{w}\alpha{-}w\beta;s),$$

where A_1, A_2 are constants . The constant A_2 must be 0, since

$$\overline{w}\alpha + w\beta \;=\; \alpha\beta\left(\frac{w}{\alpha}+\frac{\overline{w}}{\beta}\right) \;=\; -\alpha\beta\mathbf{E}\log A \;>\; 0,$$

$${}_2F_1(1+w(\alpha+\beta),1+w(\beta-\alpha);1-\overline{w}\alpha-w\beta;0) \;=\; 1$$

and $g(0)=1$. It follows that $A_1 = 1$, and, finally,

$$M_Z(s) \;=\; (1-s)\,{}_2F_1(\alpha+1,\beta+1;\overline{w}\alpha+w\beta+1;s). \tag{5}$$

Let us interpret this as a property of beta and gamma variables. The MGF of $Y = Z+C$ is

$$M_Y(s) \;=\; {}_2F_1(\alpha+1,\beta+1;\overline{w}\alpha+w\beta+1;s).$$

This is the MGF of the product of independent beta and gamma variables. To see this, suppose $\beta > \alpha$. Then

$$\overline{w}\alpha + w\beta + 1 - (\alpha+1) \;=\; w(\beta-\alpha) \;>\; 0.$$

Expanding the ${}_2F_1$ to find the moments of Y,

$$\begin{aligned}\mathbf{E}Y^n \;&=\; \frac{(\alpha+1)_n(\beta+1)_n}{(\overline{w}\alpha+w\beta+1)_n} \;=\; \frac{(\alpha+1)_n}{(\overline{w}\alpha+w\beta+1)_n}\cdot(\beta+1)_n \\ &=\; \mathbf{E}(B^{(\alpha+1,w(\beta-\alpha))})^n\cdot\mathbf{E}(G^{(\beta+1)})^n.\end{aligned}$$

Another way of writing (5) results from $Z \overset{\text{dist}}{=} AY$:

$$M_Z(s) \;=\; w\,{}_2F_1(\alpha,\beta+1;\overline{w}\alpha+w\beta+1;s) + \overline{w}\,{}_2F_1(\alpha+1,\beta;\overline{w}\alpha+w\beta+1;s). \tag{6}$$

This says that the distribution of Z is a combination of two distributions, each of which is that of the product of independent and gamma variables.

Theorem 1. *Let $\alpha,\beta>0$, $\alpha\neq\beta$, $w+\overline{w}=1$. Suppose $-\log A$ has density*

$$w\alpha e^{-\alpha x} + \overline{w}\beta e^{-\beta x}, \quad x>0,$$

and that $C\sim\mathbf{Exp}(1)$, A, C, Z independent. Then the unique solution of

$$Z \overset{dist}{=} A(C+Z)$$

has MGF (5) or (6) above. The distribution of Z is also that of

$$\sum_{n=1}^{\infty} A_1\cdots A_n C_n,$$

where $\{A_j, C_k\}$ are all independent and $A_j \overset{dist}{=} A$, $C_k \overset{dist}{=} C$.

The distribution $Y = Z+C$ is the product of independent beta and gamma variables: if $\beta>\alpha$ then

$$Y \overset{dist}{=} B^{(\alpha+1,w(\beta-\alpha))}\cdot G^{(\beta+1)}.$$

Then $AY + C \overset{dist}{=} Y$, or

$$AB^{(\alpha+1,w(\beta-\alpha))}\cdot G_1^{(\beta+1)} + G_2^{(1)} \overset{\text{dist}}{=} B^{(\alpha+1,w(\beta-\alpha))}\cdot G^{(\beta+1)}.$$

We are unable to derive this property of beta and gamma distributions from the others properties described in Dufresne (1998, 2010b), it appears to be a new identity.

By taking limits it is possible to find three other interesting formulas.

Corollary 2. *(a) If* $-\log A$ *has density*

$$(p\beta^2 x + q\beta)e^{-\beta x}, \quad x > 0, \quad 0 \le p = 1 - q \le 1,$$

(a convex combination of $\mathbf{Erlang}(2,\beta)$ *and* $\mathbf{Exp}(\beta)$ *densities) then the MGF of* Z *is*

$$(1-s)\,{}_2F_1(\beta+1,\beta+1;(1+p)\beta+1;s).$$

The distribution of $Y = Z + C$ *is that of the product of two independent* $B^{(\beta+1,p\beta)}$ *and* $G^{(\beta+1)}$*.*

(b) If A *has a* $\mathbf{Beta}(\alpha,1)$ *distribution with a point mass of probability* $p = 1-q$ *at* $u = 1$*, that is, if the distribution function of* A *is*

$$F_A(u) \;=\; \begin{cases} 0, & u \le 0 \\ qu^\alpha, & 0 < u < 1 \\ 1, & u \ge 1, \end{cases}$$

then

$$\mathbf{E}e^{sZ} \;=\; p\left(1-\frac{s}{p}\right)^{-\alpha} + q\left(1-\frac{s}{p}\right)^{-\alpha-1},$$

so the distribution of Z *is a mixture of gamma distributions with parameters* (α,p) *and* $(\alpha+1,p)$*,*

(c) If A *has a* $\mathbf{Beta}(\alpha,1)$ *distribution with a point mass of probability* $p = 1-q$ *at* $u = 0$*, that is, if the distribution function of* A *is*

$$F_A(u) \;=\; \begin{cases} 0, & u < 0 \\ p + qu^\alpha, & 0 \le u < 1 \\ 1, & u \ge 1, \end{cases}$$

then

$$\mathbf{E}e^{sZ} \;=\; p + q\cdot{}_2F_1(1,\alpha;p\alpha+1;s),$$

which says that Z *has a probability* p *of being 0, and otherwise has the same distribution as* $B^{(1,p\alpha)}\cdot G^{(\alpha+1)}$ *(these two variables being independent).*

Proof. (a) Let $0 < h < \beta$, $\alpha = \beta - h$, $w = w_h = \frac{\beta p}{h}$, $\overline{w}_h = 1 - w_h$.

Even though w_h tends to infinity as $h \downarrow 0$ the weighted exponential densities is always a true density, because

$$\begin{aligned} w_h(\beta-h)e^{-(\beta-h)x} + \overline{w}_h\beta e^{-\beta x} \;&=\; p\left[\frac{\beta}{h}(\beta-h)e^{-(\beta-h)x} + \left(1-\frac{\beta}{h}\right)\beta e^{-\beta x}\right] + q\beta e^{-\beta x} \\ &=\; p\frac{\beta}{h}(\beta-h)e^{-(\beta-h)x}(1-e^{-hx}) + q\beta e^{-\beta x} \end{aligned}$$

is positive for all $x > 0$. This expression tends to

$$(p\beta^2 x + q\beta)e^{-\beta x}$$

as h decreases to 0. At the same time

$$w_h\beta + \overline{w}_h\alpha + 1 \;=\; \frac{p\beta^2}{h} + \left(1-\frac{p\beta}{h}\right)(\beta-h) + 1 \;\to\; (1+p)\beta + 1.$$

(b) As b tend to infinity the $\mathbf{Beta}(b,1)$ tends to the degenerate distribution at 1. Hence, if $w = q, \overline{w} = p$ and one lets β tend to infinity the distribution of A becomes the one indicated.

In order to find the limit of the MGF take the limit of each term of the hypergeometric function: for $n \ge 0$,

$$\lim_{\beta\to\infty}\frac{(\alpha+1)_n(\beta+1)_n}{(p\alpha+q\beta+1)_n} \;=\; (\alpha+1)_n q^{-n}.$$

Hence,

$$\begin{aligned}\lim_{\beta\to\infty}(1-s)\,{}_2F_1(\alpha+1,\beta+1;p\alpha+q\beta+1;s) &= (1-s)\sum_{n=0}^{\infty}\frac{(\alpha+1)_n q^{-n}s^n}{n!}\\ &= (1-s)\left(1-\frac{s}{q}\right)^{-\alpha-1}\\ &= q\left(1-\frac{s}{q}\right)^{-\alpha}+p\left(1-\frac{s}{q}\right)^{-\alpha-1}.\end{aligned}$$

(c) The probability mass at 0 is obtained by letting $\overline{w}=p$, $\beta=0$, and the MGF of Z becomes

$$(1-s)\,{}_2F_1(1,\alpha+1;p\alpha+1;s).$$

The formula in the Corollary comes from (6), since ${}_2F_1(\alpha+1,0;p\alpha+1;0)=1$. □

Theorem 1 and Corollary 2 extend Proposition 3.5.3 in Dufresne (1990), which assumed A to be the product of two independent beta distributed variables:

$$A\stackrel{\text{dist}}{=}B_1^{(a,1)}\cdot B_2^{(b,1)},\quad a,b>0.$$

The density of $-\log A$ is then

$$wae^{-ax}+(1-w)be^{-bx},\quad x>0,\quad w=\frac{b}{b-a}.$$

Using Euler's formula (Lebedev, 1972, p. 248, Eq.(9.5.3)) and Theorem 1 the MGF of Z is

$$(1-s)\,{}_2F_1(a+1,b+1;a+b+1;s)\;=\;{}_2F_1(a,b;a+b+1;s),$$

which agrees with the result in Dufresne (1990).

The particular case $p=1$ in Corollary 2(a) was solved by Léveillé *et al.* (2010, Example 3.8). In this case the interarrival times T_k-T_{k-1} have an $\mathbf{Erlang}(2,\alpha)$ distribution. See Section 6.2 below for details on the MGF and distribution of $Z_t, t<\infty$.

Chamayou & Dunau (2002) look at the case $p=1,\beta=1$ of Corollary 2(a), but when C has a double exponential distribution (also known as "Laplace" distribution) $C\sim\mathbf{DoubleExp}(a)$ $(a>0)$, with density

$$\frac{a}{2}e^{-a|x|}.$$

It turns out the cases where C is exponential or double exponential can be solved in the same way, here is why. Define the $\mathbf{DoubleGamma}(a,b)$ distribution as the distribution of G_1-G_2, where $G_j\sim\mathbf{Gamma}(a,b),\ j=1,2$, are independent. (The density of G_1-G_2 may be expressed in terms of the MacDonald function, see Dufresne (1998) for details.)

Then, for $u\in\mathbf{R}$,

$$\mathbf{E}e^{iu(G_1-G_2)}\;=\;\left(\frac{b}{b-iu}\right)^a\cdot\left(\frac{b}{b+iu}\right)^a\;=\;\left(\frac{b^2}{b^2+u^2}\right)^a\;=\;\mathbf{E}\exp(-u^2G^{(a,b)}).$$

Corollary 3. *Let* $\alpha,\beta>0$, $\alpha\neq\beta$, $w+\overline{w}=1$. *Suppose* $-\log A$ *has density*

$$w\alpha e^{-\alpha x}+\overline{w}\beta e^{-\beta x},\quad x>0,$$

and that $C\sim\mathbf{DoubleExp}(1)$, *with* A,C,Z *independent. Then the unique solution of*

$$Z\stackrel{\text{dist}}{=}A(C+Z)$$

has MGF

$$\mathbf{E}e^{sZ}\;=\;w\,{}_2F_1\left(\frac{\alpha}{2},\frac{\beta}{2}+1;\overline{w}\frac{\alpha}{2}+w\frac{\beta}{2}+1;s^2\right)+\overline{w}\,{}_2F_1\left(\frac{\alpha}{2}+1,\frac{\beta}{2};\overline{w}\frac{\alpha}{2}+w\frac{\beta}{2}+1;s^2\right).$$

The variable $Y = Z+C$ has the distribution of $\sqrt{\widetilde{B}}D$, where the two variables are independent, $B \sim \mathbf{Beta}\big(\frac{\alpha}{2}+1, \frac{w}{2}(\beta-\alpha)\big)$ and $D \sim \mathbf{DoubleGamma}\big(\frac{\beta}{2}+1, 1\big)$.

Proof. The solution Z exists and is unique because $\mathbf{E}\log A < 0$ and $\mathbf{E}\log|C| < \infty$. Then, letting C_0 be an independent copy of C_k, $Y \stackrel{\text{dist}}{=} Z + C_0$, and conditioning on the $\{A_k\}$,

$$\begin{aligned} Y &\stackrel{\text{dist}}{=} C_0 + \sum_{k=1}^{\infty} A_1 \cdots A_k C_k \\ \mathbf{E}e^{iuY} &= (1+u^2)^{-1}\mathbf{E}\prod_{k=1}^{\infty}(1+u^2A_1^2\cdots A_k^2)^{-1}. \end{aligned}$$

The density of $-\log(A_1^2)$ is

$$w\frac{\alpha}{2}e^{-\frac{\alpha}{2}x} + \overline{w}\frac{\beta}{2}e^{-\frac{\beta}{2}x}, \quad x > 0.$$

Define $\widetilde{A}_k = A_k^2$, $\widetilde{C}_k = |C_k| \sim \mathbf{Exp}(1)$ and apply Theorem 1: if

$$\widetilde{Y} = \widetilde{C}_0 + \sum_{k=1}^{\infty} \widetilde{A}_1 \cdots \widetilde{A}_k \widetilde{C}_k$$

then $\widetilde{Y} \stackrel{\text{dist}}{=} \widetilde{B}\cdot\widetilde{G}$, where $\widetilde{B} \sim \mathbf{Beta}(\frac{\alpha}{2}+1, \frac{w}{2}(\beta-\alpha))$, $\widetilde{G} \sim \mathbf{Gamma}(\frac{\beta}{2}+1)$.

It follows that $\mathbf{E}\exp\left(iu\sqrt{\widetilde{B}}\left(G_1^{(\frac{\beta}{2}+1)} - G_2^{(\frac{\beta}{2}+1)}\right)\right) = \mathbf{E}e^{-u^2\widetilde{Y}} = \mathbf{E}e^{iuY}$. □

3. Second Example

Proposition 12 in Chamayou & Letac (1999) gives the solution of $Z \stackrel{\text{dist}}{=} A(Z+C)$ when A is the product of two independent $\mathbf{U}(0,1)$ variables and $C \sim \mathbf{Gamma}(2,1)$. (The density of A is then $-\log u,\ 0<u<1$.)

The solution Z has the distribution of

$$B^{(2-\phi,1+\phi)} \cdot G_1^{(2-\phi)} + G_2^{(\phi-1)},$$

where the three variables are independent, and ϕ is the "golden ratio" $\frac{1}{2}(1+\sqrt{5})$.

We extend this result to all cases where A has the distribution of the product of two independent beta variables.

Theorem 4. *Suppose $A \stackrel{\text{dist}}{=} B^{(\alpha,1)} \cdot B^{(\beta,1)}$ where the two beta variables are independent and α, β are arbitrary positive numbers; suppose also that $C \sim \mathbf{Gamma}(2,1)$. Then the unique solution of $Z \stackrel{\text{dist}}{=} A(Z+C)$ (all variables independent) has the same distribution as (assuming $\alpha \geq \beta$)*

$$B^{(\alpha+\gamma_1,\beta+\gamma_2)} \cdot G_1^{(\alpha+\gamma_1)} + G_2^{(-\gamma_1)},$$

where $\gamma_1 = \frac{1}{2}(1-\sqrt{4\alpha\beta+1})$, $\gamma_2 = \frac{1}{2}(1+\sqrt{4\alpha\beta+1})$.

The MGF of Z is

$$(1-s)^{\gamma_1}\,{}_2F_1(\alpha+\gamma_1, \beta+\gamma_1; \alpha+\beta+1; s).$$

Proof. As Chamayou and Letac, we find a differential equation for the MGF of Z, but use the method in Dufresne (1990, pp.50-51) to derive it. The identity

$$Z \stackrel{\text{dist}}{=} B_1^{(\alpha,1)} B_2^{(\beta,1)}(Z+C)$$

(with all variables independent on the right hand side) implies, by conditioning on A_1, A_2, that $M_Z(s)$ and $M_C(s)$ satisfy

$$\begin{aligned} M_Z(s) &= \int_0^1 \int_0^1 M_Z(suv) M_C(suv) \alpha\beta u^{\alpha-1} v^{\beta-1}\, du\, dv \\ &= s^{-\alpha} \int_0^s \int_0^1 M_Z(vx) M_C(vx) \alpha\beta x^{\alpha-1} v^{\alpha-1}\, dv\, dx. \end{aligned}$$

This in turn implies, after rearranging and differentiating twice,

$$\begin{aligned} sM_Z'(s) + \alpha M_Z(s) &= \alpha\beta \int_0^1 M_Z(sv) M_C(sv) v^{\beta-1}\, dv \\ &= \alpha\beta s^{-\beta} \int_0^s M_Z(y) M_C(y) y^{\beta-1}\, dv \\ sM_Z''(s) + (1+\alpha+\beta) M_Z'(s) + \frac{\alpha\beta}{s} M_Z(s)\,(1 - M_C(s)) &= 0. \end{aligned}$$

Dufresne (1990) solves this equation when C has an exponential distribution.

We now let $C \sim \mathbf{Gamma}(2,1)$, and $M_Z(s) = (1-s)^q g(s)$. From

$$\begin{aligned} M_Z' &= -q(1-s)^{q-1} g + (1-s)^q g' \\ M_Z'' &= q(q-1)(1-s)^{q-2} g - 2q(1-s)^{q-1} g' + (1-s)^q g'' \end{aligned}$$

one gets, after letting $a = \alpha + \beta + 1$ and $b = \alpha\beta$,

$$s(1-s)^2 g'' + [a(1-s)^2 - 2qs(1-s)]g' + \left[q(q-1)s - aq(1-s) + \frac{b}{s}\left((1-s)^2 - 1\right)\right] g = 0.$$

The coefficient of g reduces to

$$(q^2 - q + b)s - 2b - aq(1-s).$$

This is a multiple of $(1-s)$ if, and only if, $q^2 - q - b = 0$. Choose the smaller root $q = \gamma_1 = \frac{1}{2}(1 - \sqrt{4b+1}) < 0$. We then have

$$s(1-s)g'' + [a - (a+2q)s]g' - (aq + 2b)g = 0.$$

This is a Gauss hypergeometric differential equation

$$s(1-s)g'' + [c_3 - (c_1 + c_2 + 1)s]g' - c_1 c_2 g = 0,$$

with

$$c_1 = \alpha + q, \quad c_2 = \beta + q, \quad c_3 = \alpha + \beta + 1.$$

If c_3 is not an integer, the general solution of the differential equation is

$$g(s) = A_1\, {}_2F_1(c_1, c_2; c_3; s) + A_2 s^{1-c_3}\, {}_2F_1(1 - c_3 + c_1, 1 - c_3 + c_2; 2 - c_3; s)$$

(Lebedev, 1972, p.163).

We know that $g(0) = M_Z(0) = 1$, and so $c_3 > 1$ implies that $A_2 = 0, A_1 = 1$; thus

$$M_Z(s) = (1-s)^q\, {}_2F_1(\alpha + q, b + q; \alpha + \beta + 1; s).$$

Observe that $\alpha \geq \beta$ implies $\alpha + \frac{1}{2}(1 - \sqrt{4\alpha\beta + 1}) \geq 0$, and thus the $\mathbf{Beta}(a', b')$ distribution with parameters

$$a' = \alpha + \frac{1}{2}(1 - \sqrt{4\alpha\beta + 1}), \quad b' = b + \frac{1}{2}(1 + \sqrt{4\alpha\beta + 1})$$

exists. This gives the result when $c_3 = \alpha + \beta + 1$ is not an integer. If c_3 is an integer then a continuity argument may be used: $M_Z(-s)$ is the Laplace transform of

$$\sum_{n=1}^{\infty} A_{11} A_{21} \cdots A_{1n} A_{2n} C_n;$$

consider this Laplace transform as a function of α and β. This function is an expectation, and the Dominated Convergence Theorem implies that $M_Z(-s)$ is continuous (at least for $s > 0$) as a function of α, β. Since the hypergeometric functions found are also continuous functions of α, β, it follows that the formula obtained also applies when $\alpha + \beta + 1$ is an integer. □

Part B. Properties of Z_t, $t < \infty$

Below we consider the variable

$$Z_{S_\lambda}, \quad S_\lambda \sim \mathbf{Exp}(\lambda) \text{ independent of } \{Z_t,\, t \geq 0\}.$$

Knowing the distribution of Z_{S_λ} gives access (in theory) to the distribution of Z_t for all t and to the Laplace transform w.r.t. time

$$\int_0^\infty e^{-\lambda} \mathbf{E} g(Z_t)\, dt \;=\; \frac{1}{\lambda} \mathbf{E} g(Z_{S_\lambda})$$

(provided it exists). This is called "sampling the process at an independent exponential time S_λ". We prove a general result that connects Z_{S_λ} with the identity $X \stackrel{\text{dist}}{=} A(C + X)$.

Again suppose that: $\{T_k,\, k \geq 0\}$ are renewal times, with counting process $\{N_t,\, t \geq 0\}$, with $\mathbf{P}(T_1 > 0) = 1$, $\{C_k,\, k \geq 1\}$ is an i.i.d. sequence, independent of the renewal times, $\{R_t,\, t \geq 0\}$ is an independent Lévy process, and

$$\Sigma_t \;=\; \sum_{k=1}^{N_t} C_k, \quad Z_t \;=\; \sum_{k=1}^{N_t} e^{-R_{T_k}} C_k \;=\; \int_0^t e^{-R_{s-}}\, d\Sigma_s.$$

4. General Results

4.1. **When $\{N_t\}$ is Poisson, $\{R_t\}$ is deterministic.** If $\{T_k\}$ forms a Poisson process with parameter α the process $\{\Sigma_t\}$ is a compound Poisson (hence Lévy) process, and $R_t = rt$.

Then the characteristic function of Z_t may be found easily:

$$\mathbf{E} \exp\left(iu \int_0^t e^{-rs}\, d\Sigma_s \right) \;=\; \exp \int_0^t \Psi_\Sigma(e^{-rs} iu)\, ds$$

$$\text{where} \quad \mathbf{E} e^{u\Sigma_t} \;=\; e^{t\Psi(u)}, \quad \Psi_\Sigma(z) \;=\; \alpha(\mathbf{E} e^{zC_1} - 1).$$

("$\Psi_\Sigma(\cdot)$" is the cumulant function of the Lévy process $\{\Sigma_t\}$.) The earliest occurrence of this result that we are aware of is in Takacs (1954), in the context of shot noise (the result was rederived many times subsequently). There is no similar formula when $\{T_k\}$ is not Poisson.

4.2. **More generally.** Besides the case "N Poisson/R degenerate" (above) there are very few known explicit results about Z_t when $t < \infty$: integral equations for $\mathbf{E} e^{uZ_t}$ (Léveillé, Garrido & Wang, 2010); first and second moments of $\{Z_t\}$ have been found in a couple of specific cases; for specific numerical parameter values, density of Z_t when the inter-arrival times are $\mathbf{Erlang}(2, \beta)$ or combinations of exponentials, if C_1 has an exponential distibution (Léveillé, Garrido & Wang, 2010; Wang, Garrido & Léveillé, 2016).

Lemma 5. *If $S_\lambda \sim$ **Exp**(λ) is independent of $\{N_t\}$ then $N_{S_\lambda} \sim$ **Geom**(p), where $p = 1 - \mathbf{E} e^{-\lambda T_1}$.*

Proof. $\mathbf{P}(N_{S_\lambda} \geq n) \;=\; \mathbf{P}(S_\lambda \geq T_n) \;=\; \mathbf{E} e^{-\lambda T_n} \;=\; (\mathbf{E} e^{-\lambda T_1})^n.$ □

Let I_λ denote an indicator variable with

$$\mathbf{P}(I_\lambda = 1) \;=\; 1-p \;=\; \mathbf{P}(S_\lambda > T_1)$$
$$\mathbf{P}(I_\lambda = 0) \;=\; p \;=\; \mathbf{P}(S_\lambda \le T_1)$$

independent of $\widetilde{N}_{\widetilde{S}_\lambda}$, an independent copy of N_{S_λ}. Then

$$N_{S_\lambda} \stackrel{\text{dist}}{=} I_\lambda(1 + \widetilde{N}_{\widetilde{S}_\lambda}) \tag{7}$$

This elementary property of the geometric distribution is a particular case of other identities that follow.

Define the renewal function:

$$m(t) \;:=\; \mathbf{E}N_t \;=\; \mathbf{E}\sum_{k\ge1}\mathbf{1}_{\{T_k\le t\}} \;=\; \sum_{k\ge1}\mathbf{P}(T_k \le t).$$

Taking expectations on both sides of identity (7) yields the usual formula

$$\mathbf{E}m(S_\lambda) \;=\; \mathbf{E}N_{S_\lambda} \;=\; \frac{\mathbf{E}I_\lambda}{1-\mathbf{E}I_\lambda}, \qquad \textit{or} \qquad \int_0^\infty m(t)e^{-\lambda t}\,dt \;=\; \frac{1}{\lambda}\frac{\mathbf{E}e^{-\lambda T_1}}{1-\mathbf{E}e^{-\lambda T_1}}.$$

In the same way, recalling that $\Sigma_t \;=\; \sum_{k=1}^{N_t} C_k$,

$$\Sigma_{S_\lambda} \stackrel{\text{dist}}{=} I_\lambda(C_1 + \widetilde{\Sigma}_{\widetilde{S}_\lambda}),$$

where $\widetilde{\Sigma}_{\widetilde{S}_\lambda} \stackrel{\text{dist}}{=} \Sigma_{S_\lambda}$ and $\Sigma_{S_\lambda}, \widetilde{\Sigma}_{\widetilde{S}_\lambda}, I_\lambda$ are independent.

Since N_{S_λ} has a geometric distribution, Σ_{S_λ} has a compound geometric distribution.

Theorem 6. *If $S_\lambda \sim \mathbf{Exp}(\lambda)$ is independent of all variables in*

$$Z_t \;=\; \sum_{k=1}^{N_t} e^{-R_{T_k}}C_k,$$

(as defined above) then

$$Z_{S_\lambda} \stackrel{\text{dist}}{=} A^{(\lambda)}(C_1 + \widetilde{Z}_{\widetilde{S}_\lambda}),$$

where all variables on the right are independent and

$$A^{(\lambda)} \stackrel{\text{dist}}{=} e^{-R_{T_1}}\mathbf{1}_{\{S_\lambda > T_1\}}.$$

Moreover, $Z_{S_\lambda} \stackrel{\text{dist}}{\to} 0$ as $\lambda\to\infty$; if $Z_t \to Z_\infty$ a.s. when $t\to\infty$ then

$$Z_{S_\lambda} \xrightarrow[\lambda\downarrow 0]{\text{dist}} Z_\infty.$$

Proof. Define a new copy of Z_t, independent of T_1, C_1:

$$\widetilde{Z}_t \;=\; \sum_{k=1}^{\widetilde{N}_t} e^{-\widetilde{R}_{\widetilde{T}_k}}\widetilde{C}_k.$$

For $g(\cdot)$ bounded and measurable, on the one hand

$$\begin{aligned}\mathbf{E}g(Z_{S_\lambda}) \;&=\; \sum_{k=0}^{\infty}\mathbf{E}g\Big(\sum_{j=1}^{k}e^{-R_{T_j}}C_j\Big)\mathbf{1}_{\{T_k\le S_\lambda<T_{k+1}\}}\\ &=\; \sum_{k=0}^{\infty}\mathbf{E}g\Big(\sum_{j=1}^{k}e^{-R_{T_j}}C_j\Big)(e^{-\lambda T_k}-e^{-\lambda T_{k+1}}) \quad \text{(conditioning on } \{C_j\},\{T_k\})\end{aligned}$$

and, on the other,

$$\begin{aligned}
\mathbf{E}g(A^{(\lambda)}(C_1+\widetilde{Z}_{\widetilde{S}_\lambda})) &= g(0)\mathbf{P}(S_\lambda \le T_1) \\
&\quad + \sum_{k=0}^{\infty} \mathbf{E}g\bigg(e^{-R_{T_1}} C_1 + \sum_{j=1}^{k} e^{-(R_{T_1}+\widetilde{R}_{T_j})} \widetilde{C}_j \bigg) \mathbf{1}_{\{S_\lambda > T_1 \text{ and } \widetilde{T}_k \le \widetilde{S}_\lambda < \widetilde{T}_{k+1}\}} \\
&= g(0)\mathbf{E}(1-e^{-\lambda T_1}) \\
&\quad + \sum_{k=0}^{\infty} \mathbf{E}g\bigg(e^{-R_{T_1}} C_1 + \sum_{j=1}^{k} e^{-(R_{T_1}+\widetilde{R}_{T_j})} \widetilde{C}_j \bigg) e^{-\lambda T_1}(e^{-\lambda \widetilde{T}_k} - e^{-\lambda \widetilde{T}_{k+1}})
\end{aligned}$$

(the last step from conditioning on $C_1,\ T_1,\ \{\widetilde{C}_j\}, \{\widetilde{T}_k\}$).

Then $\mathbf{E}g(Z_{S_\lambda}) = \mathbf{E}g(A^{(\lambda)}(C_1+\widetilde{Z}_{\widetilde{S}_\lambda}))$; this can be seen by defining new sequences

$$\overline{C}_1 = C_1,\ \overline{C}_j = \widetilde{C}_{j-1},\ j \ge 2; \qquad \overline{T}_1 = T_1,\ \overline{T}_k = T_1 + \widetilde{T}_{k-1},\ k \ge 2,$$

which have the same distributions as $\{C_j,\ j \ge 1\}$ and $\{T_k,\ k \ge 1\}$, respectively.

The last claims are the usual initial and final value theorems, that are given a probabilistic flavor: S_λ has the same distribution as S_1/λ; therefore S_λ tends to 0 (resp. ∞) in probability when $\lambda \to \infty$ (resp. $\lambda \downarrow 0$). □

5. MOMENTS OF Z_t

It is a simple matter to obtain an expression for $\mathbf{E}Z_t$:

$$\begin{aligned}
\mathbf{E}\Sigma_t &= \mathbf{E}\sum_{k=1}^{N_t} C_k = \mathbf{E}N_t \cdot \mathbf{E}C_1 = m(t)\mathbf{E}C_1 \\
\mathbf{E}Z_t &= \mathbf{E}\sum_{k=1}^{N_t} e^{-R_{t_k}} C_k = \mathbf{E}\int_0^t e^{-R_s}\, d\Sigma_s = \mathbf{E}C_1 \int_0^t e^{s\Psi_R(-1)}\, dm(s).
\end{aligned}$$

Here we use the cumulant function of $\{R_t\}$, denoted $\Psi_R(\cdot)$:

$$\mathbf{E}e^{uR_t} = e^{t\Psi_R(u)}.$$

As will become clearer below, the formula for $\mathbf{E}Z^k_{S_\lambda}$ requires the existence of $\mathbf{E}C_1^k$ and of

$$\mathbf{E}(A^{(\lambda)})^k = \mathbf{E}e^{T_1(\Psi_R(-k)-\lambda)}.$$

The last expectation is finite for all $k \in \mathbf{N}$ whenever $\lambda > 0$ and either (i) $\{R_t\}$ is a subordinator, or (ii) if $\{R_t\}$ is a Brownian motion. There are of course cases where $\Psi_R(u)$ does not exist for all negative values of u, and so in those cases not all moments of Z_t are finite.

Theorem 7. *(a) Let $\rho \ge 1$. If $\mathbf{E}|C_1|^\rho < \infty$, $\mathbf{E}e^{-\rho R_t} < \infty$ then $\mathbf{E}|Z_t|^\rho < \infty$ for all $t > 0$.*

(b) Suppose $M \ge 1$ and $\mathbf{E}e^{-MR_t} < \infty$. If k is an integer, $1 \le k \le M$, then $\mathbf{E}(A^{(\lambda)})^k < 1$ for all λ large enough, and the Laplace transform of $\mathbf{E}Z_t^k$,

$$\int_0^\infty e^{-\lambda t}\mathbf{E}Z_t^k\, dt = \frac{1}{\lambda}\mathbf{E}Z^k_{S_\lambda},$$

may be found recursively from

$$
\begin{aligned}
\mathbf{E}Z_{S_\lambda} &= \frac{\mathbf{E}A^{(\lambda)}\,\mathbf{E}C_1}{1-\mathbf{E}A^{(\lambda)}} \\
\mathbf{E}Z_{S_\lambda}^2 &= \frac{\mathbf{E}(A^{(\lambda)})^2}{1-\mathbf{E}(A^{(\lambda)})^2}(2\mathbf{E}C_1\,\mathbf{E}Z_{S_\lambda}+\mathbf{E}C_1^2) \\
\mathbf{E}Z_{S_\lambda}^k &= \frac{\mathbf{E}(A^{(\lambda)})^k}{1-\mathbf{E}(A^{(\lambda)})^k}\sum_{j=0}^{k-1}\binom{k}{j}\mathbf{E}C_1^{k-j}\,\mathbf{E}Z_{S_\lambda}^j.
\end{aligned}
$$

Proof. Apply Theorem 5.1 in Vervaat (1979): suppose $\rho \geq 1$ and

$$Y \stackrel{\text{dist}}{=} AY + C, \quad Y \text{ independent of } (A,B); \tag{8}$$

if $\mathbf{E}|A|^\rho < 1$ and $\mathbf{E}|C|^\rho < \infty$, then (8) has a unique solution Y with $\mathbf{E}|Y|^\rho < \infty$.

In our case $Z_{S_\lambda} \stackrel{\text{dist}}{=} A^{(\lambda)}(Z_{S_\lambda}+C)$, which is the same as (8) if $Y = Z_{S_\lambda}+C$; the finiteness of $\mathbf{E}|Y|^\rho$ implies that of

$$\mathbf{E}|Z_{S_\lambda}|^\rho = \mathbf{E}(A^{(\lambda)})^\rho \cdot \mathbf{E}|Y|^\rho.$$

We find

$$\mathbf{E}(A^{(\lambda)})^\rho = \mathbf{E}e^{-(\lambda T_1+\rho R_{T_1})} = \mathbf{E}e^{-T_1(\lambda-\Psi_R(-\rho))}.$$

The expectation on the right is finite for $\lambda > \Psi_R(-\rho)$ and by dominated convergence tends to 0 as λ tends to infinity. The recursive equations come from raising $Z_{S_\lambda} \stackrel{\text{dist}}{=} A^{(\lambda)}(Z_{S_\lambda}+C)$ to power k.

Part (b) is obtained by raising identity in Theorem 6 to power k and then expanding. □

For instance, if the distribution of T_1 is a combination of exponentials

$$\mathbf{E}e^{-\lambda T_1} = w\cdot\frac{\alpha}{\alpha+\lambda}+\overline{w}\cdot\frac{\beta}{\beta+\lambda}$$

then $\mathbf{E}Z_{S_\lambda}$ is, as a function of λ, the ratio of a polynomial of degree 0 or 1 to a polynomial of degree 2. This implies that $\mathbf{E}Z_{S_\lambda}^2$ is the ratio of a polynomial of degree 3 (or less) to a polynomial of degree 4, and so on. This implies that $\mathbf{E}Z_t$, being the inverse Laplace transform of

$$\frac{1}{\lambda}\mathbf{E}Z_{S_\lambda}$$

is a constant plus a combination of two exponential functions $e^{-r_{1,j}t}$, where $r_{1,1}, r_{1,2}$ are the roots of $\mathbf{E}(A^{(\lambda)})-1$. Similarly, $\mathbf{E}Z_t^2$ is a constant plus a combination of four exponential functions $e^{-r_{2,j}t}$, $j=1,2,3,4$, and so on for higher moments.

5.1. **Differentiability of $\mathbf{E}e^{sZ_t}$.** Expectations of functions of Z_t satisfy differential equations, under some conditions. Obviously $\mathbf{E}Z_t$ or $\mathbf{E}e^{sZ_t}$ are not differentiable everywhere as functions of time when the distribution of T_1 has a discrete distribution. The next theorem gives sufficient conditions for those expectations to be differentiable with respect to t. We also give a general result on the value of the derivatives at $t=0$, as this is required in one of the calculation we do in the next section.

For an ordinary renewal process $\{N_t, t \geq 0\}$ those questions were solved a long time ago. For instance, if $\mathbf{E}e^{sN_t}$ is finite then, conditioning on T_1,

$$\mathbf{E}e^{sN_t} = 1-F_{T_1}(t)+\int_0^t \mathbf{E}e^{sN_{t-u}}\,dF_{T_1}(u).$$

Theorems 1 and 2 in Feller (1941) give general results about this type of renewal equation. In particular, if the distribution of T_1 is continuous and has a density which is bounded on bounded intervals then $t \mapsto \mathbf{E}e^{sN_t}$ is absolutely continuous, and therefore has a derivative with respect to t almost everywhere, and $\mathbf{E}e^{sN_t}$ is the integral of its derivative. We are going to extend this result to $\mathbf{E}e^{sZ_t}$.

Assume the waiting times $T_{j+1} - T_j$ have a continuous distribution. Then the process $\{Z_t\}$ is continuous in probability:

$$\forall\, \epsilon > 0, \quad \lim_{h \to 0} \mathbf{P}\left(|Z_{t+h} - Z_t| > \epsilon\right) \;=\; 0, \quad t \in [0, \infty).$$

If $C_j \geq 0$ then $Z_t \geq 0$ and so $\mathbf{E}e^{sZ_t}$ is finite for $\Re(s) \leq 0$; by dominated convergence the function $t \mapsto \mathbf{E}e^{sZ_t}$ is moreover continuous. In other cases we compare Z_t with

$$M_t \;=\; \sum_{j=1}^{N_t} |C_j|.$$

If $\mathbf{E}e^{|s|M_t} < \infty$ then $\mathbf{E}e^{sZ_t} < \infty$. Furthermore, it is easy to see that under the same assumptions the function $t \mapsto \mathbf{E}e^{sZ_t}$ is continuous.

By conditioning on T_1 one gets the identity:

$$\mathbf{E}e^{sZ_t} \;=\; 1 - F_{T_1}(t) + \int_0^t \mathbf{E}e^{se^{-R_u}C_1}\mathbf{E}e^{se^{-\widetilde{R}_u}Z_{t-u}}\, dF_{T_1}(u), \tag{9}$$

provided $s \in \mathbf{C}$ and all expectations exist, and if $\widetilde{R}_u$ has the same distribution as R_u but is independent of Z_{t-u}. This includes Léveillé *et al.* (2010, Lemma 2.1) as a special case (let $R_t = rt$).

Theorem 8. *Suppose $\{R_t, t \geq 0\}$ is any subordinator, that the $\{C_j\}$ are i.i.d., the waiting times $T_{j+1} - T_j$ are independent and have the same distribution on $(0, \infty)$, and that the three sets of variables $\{R_t, t \geq 0\}$, $\{C_j, j \geq 1\}$ and $\{T_j, j \geq 1\}$ are independent. Supose at least one or the following conditions holds:*

(i) $C_j \geq 0,\ s < 0$*;*

(ii) $\mathbf{E}e^{|s|M_t} < \infty,\ t \geq 0,\ s \in \mathbf{R}$.

Then:

(a) $\mathbf{E}e^{sZ_t}$ *is finite and is a continuous function of t.*

(b) If the density of T_1 is bounded on bounded intervals then $(\partial/\partial t)\mathbf{E}e^{sZ_t}$ exists for almost all t, and

$$\begin{aligned}\mathbf{E}e^{sZ_{t_1}} - \mathbf{E}e^{sZ_{t_0}} \;&=\; \int_{t_0}^{t_1} \frac{\partial}{\partial t}\mathbf{E}e^{sZ_t}\, dt, \quad 0 \leq t_0 < t_1 < \infty \\ \frac{\partial}{\partial t}\mathbf{E}e^{sZ_t} \;&=\; (\mathbf{E}e^{se^{-R_t}C_1} - 1) f_{T_1}(t) + \int_0^t \left(\frac{\partial}{\partial t}\mathbf{E}e^{se^{-\tilde{R}_u}Z_{t-u}}\right)\mathbf{E}e^{se^{-R_u}C_1} f_{T_1}(u)\, du\end{aligned} \tag{10}$$

for almost all $t \geq 0$, where $\widetilde{R}_u$ has the same distribution as R_u, and $(\widetilde{R}_u,\ Z_{t-u})$ are independent. If $f_{T_1}(0+)$ exists then

$$\lim_{t \downarrow 0} \frac{\partial}{\partial t}\mathbf{E}e^{sZ_t} \;=\; (\mathbf{E}e^{sC} - 1) f_{T_1}(0+).$$

(c) If the density of T_1 is continuous for t in some interval I then so is the function $t \mapsto (\partial/\partial t)\mathbf{E}e^{sZ_t}$.

Proof. Part (a) follows from dominated convergence. To prove (b), suppose (ii) holds, fix $0 < t_1 < \infty$, let $s \in \mathbf{R}$ and

$$G(t) \;=\; \int_0^t \mathbf{E}e^{se^{-\widetilde{R}_u}Z_{t-u}}\mathbf{E}e^{se^{-R_u}C_1}f_{T_1}(u)\,du,$$

so that $\mathbf{E}e^{sZ_t} \;=\; 1 - F_{T_1}(t) + G(t)$. By assumption $F_{T_1}(t)$ is differentiable, we need to show that $G(t)$ also is. Letting $0 < h < t,\ 0 < t + h \le t_1$,

$$\begin{aligned}|G(t+h) - G(t)| \;\le\; & \int_t^{t+h} \mathbf{E}e^{se^{-R_u}C_1}\mathbf{E}e^{se^{-\widetilde{R}_u}Z_{t+h-u}}f_{T_1}(u)\,du \\ & + \int_0^t \left|\mathbf{E}e^{se^{-\widetilde{R}_u}Z_{t+h-u}} - \mathbf{E}e^{se^{-\widetilde{R}_u}Z_{t-u}}\right|\mathbf{E}e^{se^{-R_u}C_1}f_{T_1}(u)\,du.\end{aligned}$$

Since its integrand is uniformly bounded for $0 \le u \le t + h \le t_1$, the first integral is no larger than $K_1 h$ for some constant K_1. The absolute value in the second integral is

$$|\mathbf{E}e^{se^{-\widetilde{R}_u}Z_{t-u}}(e^{se^{-\widetilde{R}_u}(Z_{t+h-u}-Z_{t-u})} - 1)| \;\le\; \mathbf{E}e^{|s|M_{t-u}}(e^{|s|(M_{t+h-u}-M_{t-u})} - 1),$$

from the inequality $|e^x - 1| \;\le\; e^{|x|} - 1$. Hence,

$$\begin{aligned}|G(t+h) - G(t)| \;&\le\; K_1 h + K_2\int_0^t (\mathbf{E}e^{|s|M_{t+h-u}} - \mathbf{E}e^{|s|M_{t-u}})\,du \\ &=\; K_1 h + K_2\int_t^{t+h}\mathbf{E}e^{|s|M_v}\,dv - K_2\int_0^h \mathbf{E}e^{|s|M_v}\,dv \\ &\le\; K_3 h,\end{aligned}$$

where K_2 is an upper bound for $\mathbf{E}e^{se^{-R_u}C_1}f_{T_1}(u)$, $0 \le u \le t_1$. This means that $G(\cdot)$ is Lipschitz continuous, hence absolutely continuous, and can be expressed as an integral:

$$G(t) \;=\; \int_0^t g(u)\,du$$

for some measurable function $g(\cdot)$. Thus $G'(t)$ exists almost everywhere, for $t \in [0, t_1]$. The number t_1 in the above argument was arbitrary, so for all $t \ge 0$ we may differentiate (9) to get (10).

Similar arguments apply when (i) holds (use the fact that Z_t is non-decreasing). □

6. DISTRIBUTION OF Z_t

6.1. **Poisson arrival times, exponential amounts.** We first confirm Theorem 6 in the case where $\{N_t\}$ is a Poisson process with parameter α, the claims are exponential with mean 1, and $R_t = rt$, $r > 0$. In this case $\{\Sigma_s\}$ is a Lévy process and as was said above the MGF of Z_t may be found from

$$\begin{aligned}\mathbf{E}\exp\left(u\int_0^t e^{-rs}\,d\Sigma_s\right) \;&=\; \exp\int_0^t \Psi_\Sigma(e^{-rs}u)\,ds \\ &=\; \exp\int_0^t \alpha\left(\frac{1}{1-ue^{-rs}} - 1\right)ds \\ &=\; \left(\frac{1-ue^{-rt}}{1-u}\right)^{\alpha/r}.\end{aligned}$$

Theorem 6 says that

$$Z_{S_\lambda} \stackrel{\text{dist}}{=} A^{(\lambda)}(C_1 + \widetilde{Z}_{\widetilde{S}_\lambda}),$$

where

$$A^{(\lambda)} \stackrel{\text{dist}}{=} e^{-rT_1}\mathbf{1}_{\{S_\lambda > T_1\}}.$$

Here T_1 is exponential with parameter α, and so (at least for $q \geq 0$)

$$\mathbf{E}(A^{(\lambda)})^q \;=\; \mathbf{E}e^{-(qr+\lambda)T_1} \;=\; \frac{\alpha}{\alpha+qr+\lambda} \;=\; \frac{\alpha}{\alpha+\lambda}\cdot\frac{\zeta}{\zeta+q},$$

where $\zeta = (\alpha+\lambda)/r$. This says that $A^{(\lambda)}$ has a probability mass $p = \lambda/(\alpha+\lambda)$ at 0, and otherwise has a $\mathbf{Beta}(\zeta, 1)$ distribution. We may then apply Corollary 2(c):

$$\mathbf{E}e^{uZ_{S_\lambda}} \;=\; (1-u)\,{}_2F_1(1, \zeta+1; p\zeta+1; u).$$

To check that the two solutions agree, apply the usual integral representation as well as a linear trnasformation for the Gauss hypergeometric function:

$$\begin{aligned}\int_0^\infty \lambda e^{-\lambda t}\left(\frac{1-ue^{-rt}}{1-u}\right)^{\alpha/r} &= \frac{\lambda}{r}(1-u)^{-\alpha/r}\int_0^1 v^{\frac{\lambda}{r}-1}(1-uv)^{\alpha/r}\,dv \\ &= (1-u)^{-\alpha/r}{}_2F_1\left(-\frac{\alpha}{r}, \frac{\lambda}{r}; \frac{\lambda}{r}+1; u\right) \\ &= (1-u)\,{}_2F_1\left(\frac{\alpha+\lambda}{r}+1, 1; \frac{\lambda}{r}+1; u\right).\end{aligned}$$

The following result, which is apparently new, shows that, while in this case the MGF of Z_t has a simple expression, the distribution of Z_t may not be so simple.

Theorem 9. *With the assumptions above, the distribution of Z_t is as follows. With probability $e^{-\alpha t}$, Z_t takes the value 0. With probability $1-e^{-\alpha t}$, Z_t has density*

$$\frac{1-e^{-rt}}{1-e^{-\alpha t}}\frac{\alpha}{r}e^{\left(1-\frac{\alpha}{r}\right)rt}\,{}_1F_1\left(1-\frac{\alpha}{r}; 2; (1-e^{rt})x\right)\,e^{-x},$$

where ${}_1F_1$ is the confluent hypergeometric function. In particular, when $\frac{\alpha}{r}$ is an integer the continuous part of the density reduces to a Laguerre polynomial times an exponential:

$$\frac{1-e^{-rt}}{1-e^{-\alpha t}}e^{\left(1-\frac{\alpha}{r}\right)rt}\;L^1_{\frac{\alpha}{r}-1}\left((1-e^{rt})x\right)\,e^{-x}.$$

Proof. The probability that Z_t equals 0 is clearly $e^{-\alpha t}$, and we know the MGF of Z_t, so it is sufficient to show that

$$\int_0^\infty e^{-ux}\cdot\beta pq^{\beta-1}e^{-x}{}_1F_1\left(1-\beta; 2; -\frac{px}{q}\right)\,dx \;=\; \left(\frac{1+qu}{1+u}\right)^\beta - q^\beta,$$

where $\beta = \frac{\alpha}{r}$, $q = e^{-rt}$, $p = 1-q$, $u \geq 0$.

First, suppose $0 < \beta < 1$. Using formulas (9.11.1) and (9.1.6) in Lebedev (1972), we find:

$$\begin{aligned}&\int_0^\infty e^{-(1+u)x}{}_1F_1\left(1-\beta; 2; -\frac{px}{q}\right)\,dx \\ &= \frac{1}{\Gamma(1-\beta)\Gamma(1+\beta)}\int_0^1 t^{-\beta}(1-t)^\beta\int_0^\infty e^{-x\left(1+u+\frac{pt}{q}\right)}\,dx\,dt \\ &= \frac{1}{\Gamma(1-\beta)\Gamma(1+\beta)(1+u)}\int_0^1 t^{-\beta}(1-t)^\beta\left(1+\frac{pt}{q(1+u)}\right)^{-1}\,dt \\ &= \frac{1}{1+u}\,{}_2F_1\left(1, 1-\beta; 2; -\frac{p}{q(1+u)}\right).\end{aligned}$$

This was proved under the assumption $0 < \beta < 1$, in order to use the integral representation for the confluent hypergeometric function. However, the first and the last expressions above are analytic functions in the region $\Re(\beta) > 0$, and the first integral converges for any β in that region, from the asymptoptic expression

$$ {}_1F_1(a;c;-z) \;\sim\; \frac{\Gamma(c)}{\Gamma(c-a)}(-z)^{-a} $$

(obtained from formulas (9.12.8) and (9.11.2) in Lebedev (1972)), which holds for $|z| \to \infty$ when $|\arg z| \le \frac{\pi}{2} - \delta$, $c, c-a \neq 0, -1, -2, \ldots$.

Next, use formula (9.2.6) in Lebedev (1972) to write

$$ {}_2F_1(1, 1-\beta; 2; z) \;=\; \frac{1-(1-z)^{\beta}}{\beta z}. $$

□

Corollary 10. *For every integer* $n \in \{0, 1, 2, \ldots\}$ *and every* $b, c > 0$ *the function*

$$ \frac{bc}{(c+1)^{n+1}-1}\, L_n^1(-bcx)\; e^{-bx}, \quad x > 0, $$

is a proper PDF, and has Laplace transform

$$ \frac{\left(\frac{c+1+u/b}{1+u/b}\right)^{n+1} - 1}{(c+1)^{n+1}-1}. $$

The $\mathbf{Exp}(b)$ *distribution is the special case* $n = 0$.

6.2. **Erlang arrival times.** Suppose waiting times follow a $\mathbf{Gamma}(2, \alpha)$ distribution, and that the $\{C_j\}$ have an exponential distribution with mean 1. The discount factor is e^{-rt}, with $r > 0$. Moments of Z_t can all be found using the techniques and results in Section 5, but finding its MGF or distribution requires more work.

Let $M_{Z_t}(s) = \mathbf{E}\exp(sZ_t)$. We use the differential equation in Corollary 3.4 of Léveillé *et al.* (2010):

$$ \frac{\partial^2}{\partial t^2} M_{Z_t}(s) = \left\{ \frac{\partial}{\partial t}\left[\log\left(\mathbf{E}e^{se^{-rt}C} - 1\right)\right] - 2\alpha \right\} \frac{\partial}{\partial t} M_{Z_t}(s) + \alpha^2 \left[M_C\left(se^{-rt}\right) - 1\right] M_{Z_t}(s). $$

Léveillé *et al.* (2010, Example 3.8) solve this in the particular case $\alpha = r = 0.01$, using the software Maple. We now give more details on the solution of this differential equation. We already know that $M_{Z_t}(s)$ is finite for $s < 1$, since Z_t is non-decreasing in t, and $\mathbf{E}\exp(sZ_\infty) < \infty$ for $s < 1$ (let $p = 1$ in Corollary 2 above). Fix $s < 1$, let $g(t) = e^{\alpha t} M_{Z_t}(s)$, and insert the MGF of C to find

$$ \left(e^{rt} - s\right) g''(t) + e^{rt} r g'(t) - e^{rt}\alpha(r+\alpha) g(t) \;=\; 0. $$

The further substitution

$$ u \;=\; \frac{2e^{rt}}{s} - 1 \quad\Longleftrightarrow\quad t \;=\frac{1}{r}\log\left[\frac{s}{2}(u+1)\right], \quad h(u) \;=\; g\left(\frac{1}{r}\log\left[\frac{s}{2}(u+1)\right]\right) $$

yields

$$ \left(1-u^2\right) h''(u) - 2uh'(u) + \nu\,(\nu+1)\, h(u) \;=\; 0, \quad \nu \;=\; \frac{\alpha}{r}. $$

This is an instance of Legendre's equation (Lebedev, 1972, p.164), which has general solution

$$ AP_\nu(u) + BQ_\nu(u), $$

where A, B are constants (that will depend on s here) and P_ν, Q_ν are respectively the Legendre functions of degree ν of the first and second kinds (Lebedev, 172, Chapter 7). To determine those constants we use the initial conditions $M_{Z_0}(s) = 1$ and

$$ \frac{\partial}{\partial t} M_{Z_t}(s)|_{t=0} = 0 $$

This initial condition is found in Léveillé *et al.* (2010, Corollary 3.4) for this special case, it is also a consequence of the more general result in Theorem 8(b) above. The value $t = 0$ corresponds to $u_0 = 2/s - 1$, and

$$h(u_0) = 1, \quad h'(u_0) = \frac{\nu s}{2}.$$

One finds

$$A = \frac{Q'_\nu(u_0) - \frac{\nu s}{2} Q_\nu(u_0)}{P_\nu(u_0)Q'_\nu(u_0) - P'_\nu(u_0)Q_\nu(u_0)}, \quad B = \frac{\frac{\nu s}{2} P_\nu(u_0) - P'_\nu(u_0)}{P_\nu(u_0)Q'_\nu(u_0) - P'_\nu(u_0)Q_\nu(u_0)}.$$

Choosing $0 < s < 1$ means that $u > 1$ for all $t \geq 0$, so the form of the Legendre functions is the one for the complex plane cut along $(-\infty, 1]$. The Wronskian of the pair $P_\nu(u), Q_\nu(u)$ is then (Lebedev, 1972, p.182):

$$P_\nu(u)Q'_\nu(u) - P'_\nu(u)Q_\nu(u) = \frac{1}{1-u^2}.$$

Inserting this into the expressions for A, B gives

$$h(u) = (1 - u_0^2)\left\{\left[Q'_\nu(u_0) - \frac{\nu s}{2} Q_\nu(u_0)\right] P_\nu(u) + \left[\frac{\nu s}{2} P_\nu(u_0) - P'_\nu(u_0)\right] Q_\nu(u)\right\},$$

or

$$\begin{aligned}\mathbf{E}e^{sZ_t} = \frac{4e^{-\alpha t}}{s^2}(1-s)&\left\{\left[\frac{\alpha s}{2r} Q_{\frac{\alpha}{r}}\left(\frac{2}{s} - 1\right) - Q'_{\frac{\alpha}{r}}\left(\frac{2}{s} - 1\right)\right] P_{\frac{\alpha}{r}}\left(\frac{2e^{rt}}{s} - 1\right)\right.\\ &\left.+ \left[P'_{\frac{\alpha}{r}}\left(\frac{2}{s} - 1\right) - \frac{\alpha s}{2r} P_{\frac{\alpha}{r}}\left(\frac{2}{s} - 1\right)\right] Q_{\frac{\alpha}{r}}\left(\frac{2e^{rt}}{s} - 1\right)\right\}.\end{aligned}$$

The cases where $\nu = \alpha/r$ is an integer are more explicit, because for $n = 0, 1, 2, \ldots$, the Legendre functions P_n, Q_n are relatively simple. The Legendre functions $P_n(\cdot)$ reduce to the Legendre polynomials:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1),$$

and so on. The Legendre functions $Q_n(\cdot)$ are (Lebedev, 1972, p.185):

$$Q_n(x) = \frac{1}{2} P_n(z) \log\frac{x+1}{x-1} - p_{n-1}(x), \quad n = 0, 1, 2, \ldots,$$

where $p_k(\cdot)$ is a polynomial.

For low values of ν it is possible to invert the MGF explicitly. For instance, if $\alpha = r$ then $\nu = 1$, $p_0(x) = 1$, which leads to

$$\begin{aligned}M_{Z_t}(s) = (1+\alpha t)e^{-\alpha t} &- \left(\frac{2}{s^2} - \frac{2 + e^{-\alpha t}}{s} + e^{-\alpha t}\right) \log\left(\frac{e^{\alpha t} - s}{1 - s}\right)\\ &+ \frac{1}{s}[2(1 - e^{-\alpha t}) - \alpha t(2 + e^{-\alpha t})] + \frac{2\alpha t}{s^2}.\end{aligned}$$

The probability that Z_t equals 0 is the first term $(1 + \alpha t)e^{-\alpha t}$ and otherwise Z_t has the density

$$2(e^{-\alpha t - e^{\alpha t}x} - e^{-x}) - \frac{e^{-x} - e^{-e^{\alpha t}x}}{xe^{\alpha t}} + (2x + 2 + e^{-\alpha t})[\mathrm{E}_1(x) - \mathrm{E}_1(e^{\alpha t}x)], \quad x > 0,$$

where $\mathrm{E}_1(z) = \int_z^\infty \frac{e^{-x}}{x}\, dx$ (the exponential integral). Letting $\alpha = r = 0.01$ in the formulas for the MGF and the density reproduce the formulas in Example 3.8 of Léveillé *et al.* (2010).

If $\frac{\alpha}{r} = 2$ then

$$\begin{aligned} M_{Z_t}(s) &= (1+\alpha t)e^{-\alpha t} + \frac{2}{s}[9(e^{-rt} - e^{-\alpha t}) - \alpha t(2e^{-\alpha t} + 3e^{-rt})] \\ &+ \frac{3}{s^2}[6e^{-\alpha t} + 4e^{-rt} - 10 + \alpha t(e^{-\alpha t} + 8e^{-rt} + 2)] \\ &+ \frac{6}{s^3}[6(1 - e^{-rt}) - \alpha t(4 + 3e^{-rt})] + \frac{18\alpha t}{s^4} \\ &+ 2\left[\frac{18}{s^4} - \frac{6}{s^3}(3e^{-rt} + 4) + \frac{3}{s^2}(e^{-\alpha t} + 8e^{-rt} + 2) - \frac{2}{s}(2e^{-\alpha t} + 3e^{-rt}) + e^{-\alpha t}\right] \\ &\times \log\left(\frac{1-s}{e^{rt} - s}\right). \end{aligned}$$

The defective density in this case is

$$\begin{aligned} &\frac{2(e^{-e^{rt}x} - e^{-x})}{xe^{\alpha t}} \\ &+ 2\Big\{3\Big[(4+2x)e^{-e^{rt}x-\alpha t} - e^{-x-\alpha t} - (3x + x^2)e^{-x} - (5+3x)e^{-x-rt} + (2+4x+x^2)e^{-e^{rt}x-rt}\Big] \\ &+ \Big[(4+3x)e^{-\alpha t} + 3(2+8x+3x^2)e^{-rt} + 3x(2+4x+x^2)]\Big)\Big][\mathrm{E}_1(x) - \mathrm{E}_1(e^{rt}x)]\Big\}. \end{aligned}$$

Routine calculations show that the limit of the MGF as t tends to infinity is

$$\frac{36}{s^3} - \frac{30}{s^2} + 12\left(\frac{3}{s^4} - \frac{4}{s^3} + \frac{1}{s^2}\right)\log(1-s).$$

This is precisely the MGF of Z_∞ as given by Corollary 2 above, since in this case rT_1 has a **Gamma**$(2, 2)$ distribution (let $p = 1, q = 0, \beta = 2$ in Corollary 2 to get

$$\mathbf{E}e^{sZ_\infty} = (1-s)\,{}_2F_1(3, 3; 5; s).)$$

7. References

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